Volume 40 July 1954 Number 7

UNIVERSITY OF MICHIGAN
JUL 21 1954
PERIODICAL READING ROOM

Proceedings of the National Academy of Sciences

of the United States of America

The Proceedings of the National Academy of Sciences

OFFICERS OF THE ACADEMY

DETLEV W. BRONK
President

GEORGE W. CORNER
Vice President

JOHN KIRKWOOD
Foreign Secretary

ALEXANDER WETMORE
Home Secretary

WILLIAM J. ROBBINS
Treasurer

EDITORIAL BOARD OF THE PROCEEDINGS

LINUS PAULING
Chairman

ALEXANDER WETMORE, *Home Secretary*
EDWIN B. WILSON, *Managing Editor*
JOHN KIRKWOOD, *Foreign Secretary*
WILLIAM W. RUBEY, *Chairman of the National Research Council*

J. W. BEAMS
R. E. CLELAND
W. W. RUBEY
HARLOW SHAPLEY
F. E. TERMAN
G. W. BEADLE
E. A. DOISY
G. C. EVANS
C. H. GRAHAM
J. T. PATTERSON
GREGORY BREIT
L. C. DUNN
W. F. LIBBY
J. VON NEUMANN
PAUL WEISS

Subscription price is $7.50 per volume (12 issues); the price of a single issue is $.75. Make all remittances payable to The University of Chicago Press in U.S. currency or its equivalent. Authorized agent for the British Commonwealth, except North America and Australasia: Cambridge University Press, Bentley House, 200 Euston Road, London, N. W. 1, England.

Editorial correspondence should be addressed to the PROCEEDINGS of the NATIONAL ACADEMY OF SCIENCES, Attention Miss Mary D. Alexander, University of Chicago Press, 5750 Ellis Avenue, Chicago 37, Illinois.

Business correspondence should be addressed to the Journals Division, University of Chicago Press, 5750 Ellis Avenue, Chicago 37, Illinois.

Subscribers are requested to notify The University of Chicago Press and their local postmaster immediately of change of address.

Microfilms of complete volumes of this journal are available to regular subscribers only and may be obtained at the end of the volume year from University Microfilms, 313 N. First Street, Ann Arbor, Michigan.

Entered as second-class matter June 21, 1920 at the Post-office at Easton, Pennsylvania, under the Act of August 24, 1912. Accepted for mailing at a special rate of postage provided for in Section 1103, Act of October 3, 1917. Authorized June 18, 1920.

PRINTED IN THE U. S. A.

THE PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES
is published for
THE NATIONAL ACADEMY OF SCIENCES
by
THE UNIVERSITY OF CHICAGO PRESS

Proceedings of the
NATIONAL ACADEMY OF SCIENCES

Volume 40 • Number 7 • July 15, 1954

FLUCTUATIONS IN THE SPACE DISTRIBUTION OF THE GALAXIES

By Vera Cooper Rubin

GEORGETOWN COLLEGE OBSERVATORY

Communicated by G. Gamow, May 3, 1954

1. *Introduction.*—In a recent paper Limber[1] has developed the equations which can be employed in an analysis of counts of galaxies to a given apparent magnitude, when the density of galaxies at a point r is described in terms of a mean density $\bar{\rho}$ and a fluctuation $D(r)$. Since the analysis requires the evaluation of the fluctuation from the mean in the number of galaxies in two directions on the sky, as a function of the angular separation of the two regions, counts of galaxies covering large continuous portions of the sky are necessary. With the exception of the data obtained by Shane,[2] no such extensive observational material is available. Hence it seems worthwhile to develop an analogous procedure which will make use of less extensive existing data.

Counts of galaxies in successive magnitude intervals are available[3] for several hundred regions of small solid angles of the sky. From these counts, the mean square fluctuation to successive magnitude limits may be calculated. The theoretical expression relating this quantity to $D(r)$ is of interest and is derived in the following section. In section 3 the observational material is analyzed and the relevant quantities calculated. By a juxtaposition of the theoretical and calculated values, the parameters which describe the fluctuating density field may be evaluated.

2. *Derivation of the Mean Square Fluctuation.*—In the following treatment, two cases will be discussed: (i) the case in which the absolute magnitude of all galaxies is equal to M_0, a constant, and (ii) the case in which the distribution of absolute magnitudes is an assumed function.

i. Let the density of galaxies in a given direction and at a distance r be given by $\rho(r)$. Then the number of galaxies within the small solid angle ω and distance r' is given by

$$N(r') = \omega \int_0^{r'} \rho(r) r^2 \, dr, \tag{1}$$

assuming space uniformly transparent. It is now assumed that the density of galaxies at any point in space may be described in terms of a mean density $\bar{\rho}$ and a fluctuation which is a function of r, $D(r)$, so that

$$\rho(r) = \bar{\rho}[1 + D(r)], \tag{2}$$

$$\overline{D(r)} = 0, \tag{3}$$

$$\overline{D^2(r)} = \beta^2 = \text{constant}, \tag{4}$$

$$\overline{D(r_1)D(r_2)} = \beta^2\Gamma(|r_1 - r_2|). \tag{5}$$

β^2 is a parameter which is equal to the mean square of the fluctuation in the density in units of the square of the density. $\Gamma(|r_1 - r_2|)$ is the correlation function of the fluctuations at the two points considered and is assumed to depend only upon the separation of the two points.

If the absolute magnitude of all galaxies is equal to M_0, then the number of galaxies to a limiting apparent magnitude m' in the small solid angle ω is

$$N(m') = \omega\bar{\rho}\int_0^A [1 + D(r)]r^2\,dr,\ {}^1/_5\,[m' + 5 - M_0]. \tag{6}$$

$$A = 10. \tag{7}$$

From equation (6),

$$\overline{N(m')} = \frac{\omega\bar{\rho}A^3}{3}, \tag{8}$$

$$\overline{N^2(m')} = \omega^2\bar{\rho}^2,\int_0^A\int_0^A [1 + \beta^2\Gamma(|r_1 - r_2|)]r_1^2r_2^2\,dr_1\,dr_2. \tag{9}$$

Define

$$\frac{\overline{\Delta N^2}}{\bar{N}^2} = \frac{\overline{N^2(m')} - \overline{N(m')}^2}{\overline{N(m')}^2}, \tag{10}$$

whence, from equations (8) and (9),

$$\frac{\overline{\Delta N^2}}{\bar{N}^2} = \frac{9\beta^2}{A^6}\int_0^A\int_0^A \Gamma(|r_1 - r_2|)r_1^2r_2^2\,dr_1\,dr_2. \tag{11}$$

For each of the two forms of the correlation function,

a) $$\Gamma(|r_1 - r_2|) = e^{-|r_1 - r_2|/r_0}, \tag{12}$$

b) $$\Gamma(|r_1 - r_2|) = e^{-|r_1 - r_2|^2/r_0^2}, \tag{13}$$

equation (11) may be integrated twice to obtain, for *a*),

$$\frac{\overline{\Delta N^2}}{\bar{N}^2} = 9\beta^2\left[e^{-A/r_0}\left\{\frac{4r_0^4}{A^4} + \frac{8r_0^5}{A^5} + \frac{8r_0^6}{A^6}\right\} + \frac{2r_0}{5A} - \frac{8r_0^6}{A^6} - \frac{r_0^2}{A^2} + \frac{4r_0^3}{3A^3}\right]. \tag{14}$$

For $A \gg r_0$,

$$\frac{\overline{\Delta N^2}}{\bar{N}^2\beta^2} \cong \frac{18}{5}\frac{r_0}{A}. \tag{14a}$$

For *b*,

$$\frac{\overline{\Delta N^2}}{\bar{N}^2} = 9\beta^2\left[-\frac{r_0^6}{15A^6} - \frac{r_0^2}{2A^2} + \left(\frac{\sqrt{\pi}\,r_0}{5A} + \frac{\sqrt{\pi}r_0^3}{6A^3}\right)\Phi\left(\frac{A}{r_0}\right) + \left(\frac{r_0^6}{15A^6} + \frac{r_0^4}{15A^4} + \frac{r_0^2}{5A^2}\right)e^{-A^2/r_0^2}\right], \tag{15}$$

$$\Phi\left(\frac{A}{r_0}\right) = \frac{2}{\sqrt{\pi}} \int_0^{A/r_0} e^{-A^2/r_0^2}\, d\left(\frac{A}{r_0}\right). \tag{16}$$

For $A \gg r_0$,

$$\frac{\overline{\Delta N^2}}{\bar{N}^2\beta^2} \cong \frac{9}{5}\frac{\sqrt{\pi} r_0}{A}. \tag{16a}$$

r_0 is the microscale of the density distribution. In Figure 1, equations (14) and (15) are plotted as a function of A/r_0.

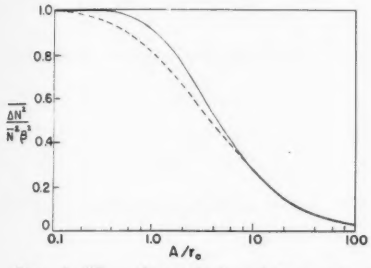


FIG. 1. The theoretical relative mean square fluctuation $\overline{\Delta N^2}/\bar{N}^2\beta^2$ in the number of galaxies counted to limiting magnitude m' for the case $\Gamma = e^{-r/r_0}$ (*dashed line*) and $\Gamma = e^{-r^2/r_0^2}$ (*solid line*); $^1/_5[m' + 5 - M_0]$ $A = 10$.

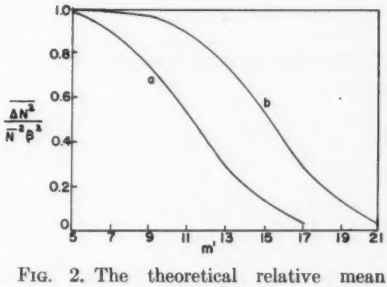


FIG. 2. The theoretical relative mean square fluctuation $\overline{\Delta N^2}/\bar{N}^2\beta^2$ in the number of galaxies counted to limiting magnitude m' for the case $\Gamma = e^{-r/r_0}$ and an assumed luminosity function for the galaxies, $N(M) =$ constant, $M_1 \geq M \geq M_2$. (*a*) $r_0 = 2 \times 10^6$ parsecs; (*b*) $r_0 = 1.26 \times 10^7$ parsecs.

ii. Assume a luminosity function in which $N(m)$ is a constant, $M_1 \geq M \geq M_2$, and 0 outside this range. Let

$$a = 10^{\frac{+5 - M_1}{5}}, \tag{17}$$

$$b = 10^{\frac{m' + 5 - M_2}{5}}. \tag{18}$$

Under the above assumptions, the number of galaxies to a limiting apparent magnitude

$$N(m') = \frac{\omega\bar{\rho}}{M_1 - M_2} \int_{M_1}^{M_2} dM \int_0^{r'} r^2\, dr\, [1 + D(r)] \tag{19}$$

may be written

$$N(m') = G \int_a^b \frac{dr'}{r^1} \int_0^{r'} r^2\, dr\, [1 + D(r)], \tag{20}$$

$$G = \frac{\omega\bar{\rho}5 \log_{10} e}{M_1 - M_2}. \tag{21}$$

From equation (20),

$$\bar{N} = \frac{G}{9}[b^3 - a^3], \tag{22}$$

$$\overline{\Delta N^2} = G^2 \int_a^b \int_a^b \frac{dr_1'}{r_1'} \frac{dr_2'}{r_2'} \int_0^{r_1'} \int_0^{r_2'} r_1^2 r_2^2\, dr_1 dr_2\, \overline{D(r_1)D(r_2)}. \tag{23}$$

Inserting the expression

$$\Gamma = e^{-|r_1 - r_2|/r_0} \tag{24}$$

and integrating first over r_1 and then over r_2, equation (24) becomes

$$\frac{\overline{\Delta N^2}}{\bar{N}^2} = \frac{9^2\beta^2}{[b^3 - a^3]^2} \int_a^b \int_a^b \frac{dr_1'}{r_1'} \frac{dr_2'}{r_2'} \left\{ \frac{2r_0 r_2'^5}{5} + \frac{4r_0^3 r_2'^3}{3} + 2r_0^3 e^{-r_2'/r_0} \times \right.$$
$$[r_2'^2 r_0 + 2r_2' r_0^2 + 2r_0^3] - 4r_0^6 + e^{-r_1'/r_0} e^{r_2'/r_0} [-r_0 r_1'^2 - 2r_0^2 r_1' - 2r_0^3] \times$$
$$\left. [r_2'^2 r_0 - 2r_2' r_0^2 + 2r_0^3] + 2r_0^3 e^{-r_1'/r_0} [r_0 r_1'^2 + 2r_0^2 r_1' + 2r_0^3] \right\}. \tag{25}$$

Because the integrand is symmetrical in r_1' and r_2', this may be written, dropping primes,

$$\frac{\overline{\Delta N^2}}{\bar{N}^2} = \frac{2 \cdot 9^2\beta^2}{[b^3 - a^3]^2} \int_a^b \frac{dr_1}{r_1} \int_a^{r_1} \left(\frac{2r_0 r_2^4}{5} + \frac{4r_0^3 r_2^2}{3} + 2r_0^3 e^{-r_2/r_0} \times \right.$$
$$\left[r_2 r_0 + 2r_0^2 + \frac{2r_0^3}{r_2} \right] - \frac{4r_0^6}{r_2} + e^{-r_1/r_0} e^{r_2/r_0} [-r_0 r_1^2 - 2r_0^2 r_1 - 2r_0^3] \times$$
$$\left. \left[r_2 r_0 - 2r_0^2 + \frac{2r_0^3}{r_2} \right] + \frac{2r_0^3}{r_2} e^{-r_1/r_0} [r_0 r_1^2 + 2r_0^2 r_1 + 2r_0^3] \right) dr_2 \tag{26}$$

After two integrations, this becomes

$$\frac{\overline{\Delta N^2}}{\bar{N}^2} = \frac{2 \cdot 9^2 r_0^6 \beta^2}{[b^3 - a^3]^2} \left\{ \left[4k + 2e^{a/r_0} \left(\frac{a}{r_0} - 3 \right) \right] \int_{a/r_0}^{b/r_0} \frac{e^{-x}}{x} dx + \right.$$
$$e^{-b/r_0} \left(\frac{b}{r_0} + 3 \right) \left(2 \int_{a/r_0}^{b/r_0} \frac{e^x}{x} dx - e^{a/r_0} \left[\frac{a}{r_0} - 3 \right] - 2k \right) +$$
$$e^{-a/r_0} \left[e^{a/r_0} \left(\frac{a}{r_0} + 3 \right) \left(\frac{a}{r_0} - 3 \right) + 2k \left(\frac{a}{r_0} + 3 \right) \right] - 2k^2 + 2 \left(\frac{b}{r_0} - \frac{a}{r_0} \right) +$$
$$\frac{1}{2} \left(\frac{b^2}{r_0^2} - \frac{a^2}{r_0^2} \right) + \frac{1}{9} \left[-\frac{5}{3} \left(\frac{b^3}{r_0^3} - \frac{a^3}{r_0^3} \right) - \frac{4a^3 k}{r_0^3} \right] + \frac{2}{25} \left[\frac{1}{5} \left(\frac{b^5}{r_0^5} - \frac{a^5}{r_0^5} \right) - \right.$$
$$\left. \left. \frac{a^5}{r_0^5} k \right] - 4 \int_{a/r_0}^{b/r_0} \frac{e^{-y}}{y} \int_{a/r_0}^{y} \frac{e^x}{x} dx \, dy \right\}. \tag{27}$$

$$k = \log_e \frac{b}{a} = (M_1 - M_2) \frac{\log_e 10}{5}. \tag{28}$$

The last term of equation (27) has been evaluated numerically. Equation (27) is plotted in Figure 2 as a function of limiting apparent magnitude m', for two sets of values:

$$M_1 = -11.5, \quad M_2 = -18.0, \quad r_0 = 2 \times 10^6 \text{ parsecs},$$
$$M_1 = -11.5, \quad M_2 = -18.0, \quad r_0 = 1.26 \times 10^7 \text{ parsecs}.$$

The limits of the luminosity function have been determined so that the resulting luminosity function corresponds approximately to Holmberg's[4] luminosity function, which has been revised for the new distance scale.

It may be noted that as $a \rightarrow b$ in equation (27), i.e., the limits of the luminosity function approach M_0, the expression goes over to equation (14), the analogous equation derived with the assumption that $M = M_0$ for all galaxies.

3. *Observational Material.*—From publications of the Harvard College Observatory, one hundred plates are available[5] with $|b|$ over 30°. For each plate the number of galaxies at each one-tenth magnitude interval for the central 9 square degrees of the plate has been tabulated.[6] The distribution on the sky of these plates is shown in Figure 3.

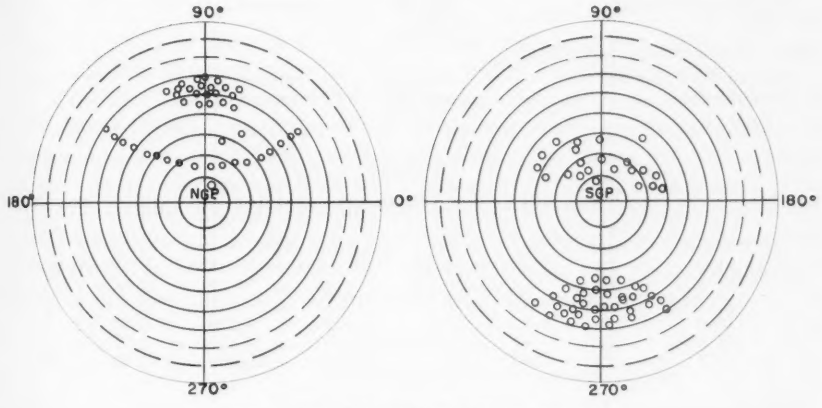


FIG. 3 Distribution of 100 Harvard plates

The following corrections have been applied to the published data.

a) The faint end of the Harvard magnitude scale appears systematically too bright by several tenths of a magnitude. This discrepancy, noted previously by Seyfert[7] by comparing Harvard magnitudes with measures obtained by Baade,[8] is further indicated by the fact that the logarithm of the ratio of the number of galaxies at one magnitude to that at the previous, log $[A(m + 1)/A(m)] = 0.70$, which is considerably higher than the theoretical value for uniform space distribution. While compatible with the Stebbins-Whitford effect, the amount of the excess is too great to be accounted for by an increase in the mean absolute magnitude of galaxies with increasing distance from our galaxy. Therefore, each observed magnitude m_0 has been corrected by an amount $\Delta m = \frac{(m_0 - 14.7 \pm 0.10)}{5.7}$. The mean error has been estimated.

b) The magnitude readings of each plate have been corrected by an amount $\Delta m = \kappa \csc \beta/0.6$ for the effects of interstellar absorption, and the resulting magnitude of each galaxy is that which would be observed outside the obscuration within our galaxy. From the number of galaxies up to magnitude 17.5 for each of the 100 plates, κ was determined as $\kappa = 0.38 \pm 0.05$.

c) No corrections for differential extinction or plate limb effects are included, for studies by Shapley and his co-workers[9] have shown such effects to be negligible.

No red-shift corrections are applied, for the amount of such correction is negligible at the distances included in this survey.

The over-all mean error is estimated from Table 1. The counted number of galaxies N to a limiting magnitude has a mean error $\pm N/3$.

TABLE 1

	Amount (M.E.)	Reference
Source of error in magnitude:		
Individual magnitude estimate	$\pm 0^m.17$	*HA* **105**, No. 10; *HB* **895, 905**; *HC* **423.**
Magnitude scale correction	± 0.10	Estimated
Absorption correction	± 0.085	Calculated
Total	± 0.21	
Source of error in counts of N galaxies:		
Magnitude error $\pm 0^m.21$	$\pm 0.33N$	Calculated
Counting error	$\pm 0.05N$	Estimated
Total	$\pm 0.33N$	

TABLE 2

VALUES OF $\overline{N}$, $\overline{\Delta N^2}$, AND $\overline{\Delta N^2}/\overline{N^2}$ FOR THE HARVARD PLATES

$m'_{corrected}$	No. Plates	$\overline{N}$	$\overline{\Delta N^2}$	$\overline{\Delta N^2}/\overline{N^2}$
14.2	82	3.4 ± 0.18	13.3 ± 4.0	1.15 ± 0.34
14.8	100	8.2 ± 0.34	40 ± 12	0.60 ± 0.18
15.4	100	18.3 ± 0.71	135 ± 47	0.40 ± 0.14
16.6	99	92 ± 3.6	$3{,}580 \pm 1{,}470$	0.42 ± 0.17
17.3	65	181 ± 8.3	$8{,}790 \pm 4{,}650$	0.27 ± 0.14
17.7	21	$282 \pm 23.$	$23{,}690 \pm 21{,}000$	0.30 ± 0.26

The homogeneity of the observational material may be examined as follows. The 100 plates are arranged according to latitude, and the mean number of galaxies at any magnitude (after correction) is determined by including counts from first

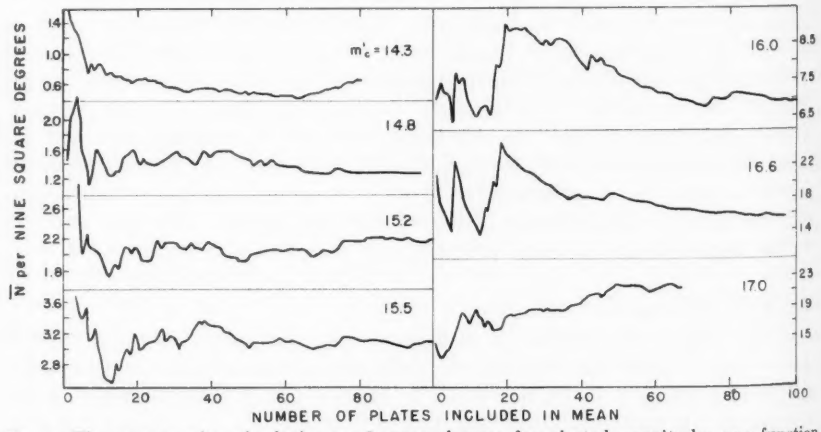


FIG. 4. The mean number of galaxies, per 9 square degrees, for selected magnitudes, as a function of the number of plates included in the mean.

one plate in the mean, then two plates, and so on, until counts for all plates are included in the mean. A plot of the mean number of galaxies against the number of

plates included in the mean indicates a leveling-off, at all magnitudes, when about 60 plates have been included. Some representative plots are shown in Figure 4. Thus it appears valid to group the data for the 100 plates without any loss of information.

Table 2 includes values of $\bar{N}$, $\overline{\Delta N^2}$, and $\overline{\Delta N^2}/\bar{N}^2$, and their mean errors. All values have been calculated from the counts for the central 9 square degrees of each plate.

In addition to the above material, counts of the total number of galaxies recorded to the magnitude limit of the plate are available[10] for 222 plates in the two zones; $57° \leq |b| \leq 75°$, $76° \leq |b| \leq 90°$. Table 3 includes this available material. The first column is the source of the data, and the second column the arbitrary magnitude limit to which all counts have been reduced by Shapley, on the assumption of uniform space density. The third column contains the latitude ranges into which the data have been grouped for the present analysis, with the absorption correction assumed to be constant over the zone. The fourth and fifth columns list the corrected magnitude limit and the number of plates in each zone. The final three columns contain values of $\bar{N}$, $\overline{\Delta N^2}$, and $\overline{\Delta N^2}/\bar{N}^2$, per square degree, calculated from counts of the central 9 square degrees of each plate, to the noted magnitude limit. The final entry refers to the data of Shane, with the limiting magnitude 18.6 listed by Neyman and Scott reduced to 18.0 outside the galaxy.

TABLE 3

VALUES OF $\bar{N}$, $\overline{\Delta N^2}$, AND $\overline{\Delta N^2}/\bar{N}^2$ FOR THE REMAINING PLATES

Reference	m'_{reduced}	Latitude	$m'_{\text{corrected}}$	No. Plates	$\bar{N}$	$\overline{\Delta N^2}$	$\overline{\Delta N^2}/\bar{N}^2$
HR 347	17.5	58°–75°	17.3	100	44.8 ± 1.6	470 ± 218	0.27 ± 0.11
HR 347	17.5	76 –90	17.4	23	47.5 ± 3.6	477 ± 499	$.21 \pm .22$
HR 333	17.9	57 –75	17.8	27	28.1 ± 1.9	134 ± 152	$.17 \pm .19$
HC 423	18.2	58 –75	18.1	53*	53.9 ± 2.8	$1,017 \pm 585$	$.35 \pm .20$
HC 423	18.2	76 –90	18.2	19	39.4 ± 3.2	246 ± 320	$.16 \pm .20$
Astrophys. J. **117**, 132, 1953	..	...	18.0	5,796 sq. degrees	89	1,200	0.15

* When three high-density plates are omitted, $\bar{N} = 48.1$, $\overline{\Delta N^2}/\bar{N}^2 = 0.21$.

In Figures 5 and 6, the calculated values of $\overline{\Delta N^2}/\bar{N}^2$ are superimposed on the theoretical curve for $\overline{\Delta N^2}/\bar{N}^2\beta^2$, $\beta^2 = 1$. In Figure 5 the abscissae are $10^{m'/5}$ (*upper*) and A/r_0 (*lower*) for the calculated and theoretical values, respectively. With $\beta^2 = 1$, the following values may be read from the graphs:

$$\text{For } \Gamma = e^{-r/r_0}\text{: at } A/r_0 = 1;\ 10^{m'/5} = 2.5 \times 10^2, \text{ or } m' = 12.0.$$

$$\text{For } \Gamma = e^{-r^2/r_0^2}\text{: at } A/r_0 = 1;\ 10^{m'/5} = 2.5 \times 10^2, \text{ or } m' = 12.0.$$

Thus r_0, the microscale of the fluctuations, is equal to the mean distance of a twelfth-magnitude galaxy, independent of any assumption of the mean absolute magnitude of a galaxy. If $M_0 = -14.8$, $r_0 = 2.3 \times 10^6$ parsecs.

In Figure 6, $\Gamma = e^{-r/r_0}$, $M_1 = -11.5$, $M_2 = -18.0$, $r_0 = 1.26 \times 10^7$ parsecs.

As β^2 increases, the calculated values will fall on the tail of the curve, where $A/r_0 > 1$. Due to the asymptotic form of the expressions for $\overline{\Delta N^2}/\bar{N}^2\beta^2$ [equations (14*a*) and (16*a*)], it may be seen that the variation of r_0 with β^2 is given by $r_0\beta^2 = 2.3 \times 10^6$ parsecs, or $r_0\beta^2 = 1.26 \times 10^7$ parsecs, $\beta^2 \geq 1$. It appears that the inclu-

sion of a luminosity function affects in a significant way the microscale of the fluctuations.

This analysis does not permit independent determinations of r_0 and β^2. However, under certain conditions, estimates of r_0 may be made on physical grounds,[10] and β^2 evaluated.[11]

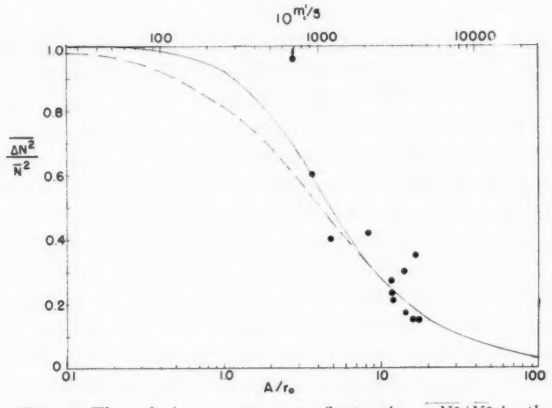


FIG. 5. The relative mean-square fluctuation $\overline{\Delta N^2}/\bar{N}^2$ in the number of galaxies counted to limiting magnitude m'_c for all available data, as a function of $10^{m'_c/5}$ (*upper scale*). By superimposing the observed values on the predicted curves, the quantity $r_0\beta^2$ may be evaluated. The dashed curve is derived for $\Gamma = e^{-r/r_0}$, $\beta^2 = 1$, while the solid curve is derived for $\Gamma = e^{-r^2/r_0^2}$, $\beta^2 = 1$.

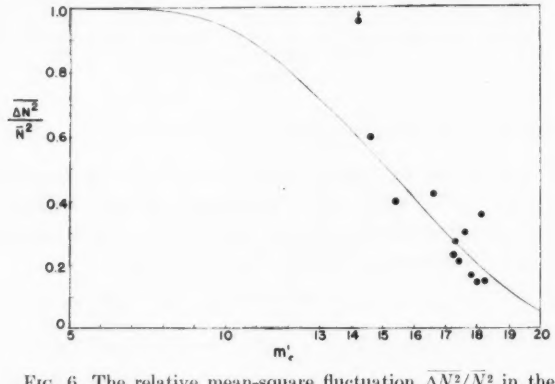


FIG. 6. The relative mean-square fluctuation $\overline{\Delta N^2}/\bar{N}^2$ in the number of galaxies counted to limiting magnitude m'_c for all available data, as a function of m'_c. The curve represents the predicted values with $r_0 = 1.26 \times 10^7$ parsecs, $\beta^2 = 1$.

4. *Conclusions.*—It is apparent that the space distribution of the galaxies may be analyzed in terms of a fluctuating density field, using methods developed from the physical theory of turbulence. The advantages of the method are twofold: it requires a minimum number of parameters, and it is physically reasonable if the

galaxies have condensed from a turbulent gaseous medium. An upper limit to r_0, the microscale of the turbulence, is of the order of 10^7 parsecs and hence is of the order of the dimensions of clusters of galaxies.

I would like to express my thanks to Professor Gamow, for suggesting this investigation and for his continued enthusiasm throughout its course; to Dr. Frenkiel for helpful discussions on the theory of turbulence and for suggestions concerning the treatment of the data; to Fr. Heyden and Dr. Shapley for their continued interest in the problem; to Dr. Baade for a discussion of the magnitude error of the Harvard plates; and to Dr. Robert J. Rubin, with whom all phases of this work have been discussed.

[1] D. N. Limber, *Astrophys. J.*, **117**, 134–144, 1953.

[2] C. D. Shane and C. A. Wirtanen, *Proc. Am. Phil. Soc.*, **94**, 13–17, 1950; J. Neyman, E. L. Scott, and C. D. Shane, *Astrophys. J.*, **117**, 92–133, 1953.

[3] *Harvard Ann.*, **105**, No. 8, 1937; **105**, No. 10, 1937; **106**, No. 1, 1938; these PROCEEDINGS, **26**, 166, 1940; **26**, 554, 1940; **36**, 157, 1950; **37**, 191, 1951; *Harvard Circ.*, No. 423, 1937.

[4] E. Holmberg, *Lund. Medd.*, ser. II, p. 128, 1950.

[5] *Harvard Ann.*, **105**, No. 8; **105**, No. 10; **106**, No. 1; *Harvard Reprints*, Nos. 194, 208.

[6] *Harvard Reprint* No. 208 totals frequencies for the 25 central square degrees. Shapley has, advised, in a private communication, that the tabulated values merely be reduced by the factor 9/25.

[7] C. Seyfert, *Harvard Bull.*, **905**, 22, 1937.

[8] W. Baade, *Astrophys. Nachr.*, **233**, 65, 1928.

[9] *Harvard Ann.*, **106**, No. 1, 7, 1938.

[10] G. Gamow, these PROCEEDINGS, **40**, 480–484, 1954.

[11] The author is indebted to Dr. Chandrasekhar for a copy of a paper by Limber to be published in the May, 1954, *Astrophysical Journal*, in which the data of Shane are analyzed in terms of a fluctuating density field. Limber's results, while more complete than those obtained here, indicate values of the same order of magnitude as the above.

---

*MAGELLANIC CLOUDS. XIII. COMPARISON OF MAGELLANIC AND GALACTIC ECLIPSING VARIABLES**

BY HENRY NORRIS RUSSELL

PRINCETON UNIVERSITY OBSERVATORY

Communicated April 15, 1954

Mrs. Nail's observations[1] of eclipsing stars in the Magellanic Clouds provide conclusive evidence that such systems are numerous there and make it possible, for the first time, to compare them with objects of the same type in the Galaxy.

Periods, apparent photographic magnitudes, and ranges are available for forty-four Magellanic pairs. With the distance modulus 19.0[2] (uncorrected for space absorption), the absolute magnitudes at maximum run from -6.45 to -2.5. The spectra of the variables have not been observed, but Shapley and Nail are convinced that they are blue stars, presumably of early B type.

Galactic eclipsing variables of comparable absolute magnitude must be selected for comparison. This must be done from the available data regarding linear diameter and surface brightness.

The best available list appeared to be an old one[3] which was at least impartially selected and complete at the time of its publication. After the manuscript of the present study was sent to England, S. Gaposchkin's extensive list[4] of elements of eclipsing variables appeared and provided material for an independent comparison by the selection from his list of all pairs, except those with ranges less than $0^m.5$, for which the combined light (photographic) of the components is brighter than $M = -2.0$. Three stars—RX Cassiopeiae, S Doradus, and W Serpentis—for which the published light-curves are quite aberrant, were also omitted; and S Doradus, a Magellanic star, was excluded by Shapley and Nail.

These two differently selected—though not exclusive—groups of galactic eclipsing variables are available for comparison with the Magellanic stars. Their distribution with respect to period and absolute magnitude is as shown in Table 1.

TABLE 1

DISTRIBUTION OF ECLIPSING VARIABLES BRIGHTER THAN $M = -2$

	Period (Days)					
	1–2	2–4	4–8	8–16	16–200	All
Magellanic	14	16	10	2	2	44
Galactic (R. and M.)	6	5	1	0	0	12
Galactic (S. G.)	9	10	9	7	4	39

	Absolute Magnitude						
	−2 to −3	−3 to −4	−4 to −5	−5 to −6	−6 to −7	−7 to −8	All
Magellanic	7	20	12	4	1	..	44
Galactic (R. and M.)	5	2	1	2	0	1	11
Galactic (S. G.)	21	7	8	1	1	1	39

No data are given for Magellanic stars brighter than $12^m.5$ ($M = -6.5$), since they could be distinguished from foreground stars only by their radial velocities. The five galactic variables in Gaposchkin's list with periods between 15 and 198 days have spectra ranging from B2 to G6. The galactic survey by S. and C. P. Gaposchkin was very thorough to the tenth magnitude, and Mrs. Nail concludes that fully two-thirds of the eclipsing variables in the Small Cloud have been found—though a smaller proportion in the Large Cloud—so that no considerable increase in the data is soon to be expected.

The conclusion that, so far as these general properties are concerned, the Magellanic and galactic eclipsing variables are strikingly similar appears to be well established. This is by no means surprising, but it deserves explicit statement. From comparison of Mrs. Nail's and Gaposchkin's lists the evidence appears to be conclusive that the number of eclipsing variables brighter than $M = -2$ in the Clouds considerably exceeds that of the eclipsing variables of similar brightness in the Galaxy within 2500 psc. of the sun, without allowance for space absorption in either case.

* The results here summarized will appear in detail in a volume to be published by the Cambridge Observatories in honor of Dr. F. J. M. Stratton.

[1] H. Shapley and V. McK. Nail, these PROCEEDINGS, **39**, 1–5, 1953; *Harvard Reprints*, No. 368.

[2] Personal communication from Dr. Shapley.

[3] H. N. Russell and C. E. Moore, *The Masses of the Stars* (Chicago: University of Chicago Press, 1940), p. 85.

[4] S. Gaposchkin, *Harvard Ann.*, Vol. **113**, No. 2, 1953.

THE DIFFUSION COEFFICIENTS OF THE ALKALI METAL CHLORIDES AND POTASSIUM AND SILVER NITRATES IN DILUTE AQUEOUS SOLUTIONS AT 25°*

By Herbert S. Harned

DEPARTMENT OF CHEMISTRY, YALE UNIVERSITY

Communicated April 26, 1954

Recently,[1, 2] the determination of diffusion coefficients of electrolytes in dilute aqueous solutions has been made possible by the utilization of electrolytic conductance for the required determination of concentration at suitable intervals of time. This method is the only one which yields accurate determinations in the very dilute solution range, and therefore the results derived from it are suitable for testing the Onsager and Fuoss[3] equations for the concentration dependence of electrolytic diffusion.

Since the method of measurement has been described in detail elsewhere,[1, 2] only an outline sufficient to define the quantities involved and the calculations will be presented in the sequel. Emphasis will be placed on the actual experimental results as related to theory, and the discussion will be confined to uniunivalent electrolytes in the concentration range from 0.0005 to 0.01 molar.

Theoretical Considerations.—Fick's first law of diffusion may be expressed by

$$\mathbf{J} = c\mathbf{v} = -\mathfrak{D}\nabla c = -\overline{\mathfrak{M}}\nabla\mu, \tag{1}$$

where $\mathbf{J}$ is the flow, c the concentration of electrolyte, ∇ the operator del, $\mathbf{v}$ the velocity of flow, and $\mathfrak{D}$ the diffusion coefficient. The theory of irreversible processes[4] for mass flow at constant temperature and pressure requires the proportionality of the flow to the gradient of the chemical potential, $\nabla\mu$, as represented by the term on the right of equation (1). It follows that

$$\mathfrak{D} = \left(\frac{\partial\mu}{\partial c}\right)_{p,\,T} \overline{\mathfrak{M}}. \tag{2}$$

If c is expressed in moles per liter, and the equation which related the mean activity coefficient of the electrolyte $y_\pm$ on the molar concentration scale to the relative chemical potential is introduced in equation (2),

$$\mathfrak{D} = \nu\, 1000\, RT\, \frac{\overline{\mathfrak{M}}}{c}\left(1 + c\,\frac{\partial \ln y_\pm}{\partial c}\right) \tag{3}$$

is obtained, where ν is the number of ions into which the electrolyte dissociates, R is the gas constant, and T the absolute temperature. The term in parentheses, which results from the gradient of the chemical potential, is denoted the "thermodynamic term," and $(\overline{\mathfrak{M}}/c)$ the "mobility term."

In dilute solutions, where factors such as change in viscosity may be neglected, the concentration dependence of the mobility has been evaluated by Onsager and Fuoss.[3, 5] When an electrolyte diffuses into a solvent, both ions travel with the same velocity, as required by electrical neutrality. In this case, electrophoresis occurs because the ions which move in one direction are replaced by solvent molecules which move in the opposite direction. By the requirement that the forces on

the ions must be balanced by compensating forces on the water molecules, by assuming Stokes's law, which relates the velocity with these forces, and by utilizing the Debye and Hückel[6] potential of the ion and its atmosphere in the Maxwell-Boltzmann equation for the number of ions of one kind in the presence of another ion, Onsager and Fuoss obtained the following expression for the mobility term:

$$\frac{\overline{\mathfrak{M}}}{(c)} = 1.0748 \times 10^{-20} \frac{\lambda_1^0\lambda_2^0}{\nu_1|z_1|\Lambda^0} - \frac{(|z_2|\lambda_1^0 - |z_1|\lambda_2^0)^2}{|z_1z_2|(\nu_1 + \nu_2)\Lambda^{02}} \frac{3.122 \times 10^{-19}}{\eta_0(DT)^{1/2}} \times \frac{\sqrt{\Gamma}}{(1 + \kappa a)} + \left(\frac{z_2^2\lambda_1^0 + z_1^2\lambda_2^0}{\Lambda^0}\right)^2 \frac{9.304 \times 10^{-13}}{\eta_0(DT)^{1/2}} c\phi(\kappa a). \quad (4)$$

Here λ_1^0 and λ_2^0 are the equivalent cation and anion conductances, Λ^0 their sum, z_1 and z_2 their valences, ν_1 and ν_2 the number of anions and cations into which the electrolyte dissociates, η_0 the viscosity of the solvent, D its dielectric constant, Γ the ional concentration or $\Sigma c_i z_i^2$, κ the Debye and Hückel reciprocal radius, and a the distance parameter of the order of magnitude of the sum of the crystal radii of the electrolyte. The quantity $\phi(\kappa a)$ is the exponential integral function of the theory, values of which have been tabulated.[7]

It is to be noted that the thermodynamic term equals unity and the mobility term equals the concentration-independent members on the right when c equals zero. Consequently, equations (3) and (4) yield the limiting equation for the diffusion coefficient $\mathfrak{D}_0$ originally obtained by Nernst,[8] which, for uniunivalent electrolytes in water at 25°, becomes

$$\mathfrak{D}_0 = 17.872 \times 10^{-10} T \left(\frac{\lambda_1^0\lambda_2^0}{\Lambda^0}\right). \quad (5)$$

The electrophoretic contribution is given by the concentration-dependent terms of equation (4). It is to be noted that these terms have opposite signs. Indeed, for electrolytes such as lithium chloride, in which the cation and anion mobilities differ considerably, the magnitudes of the positive and negative terms are about equal, and their sum is quite small. In this case the concentration dependence of the diffusion coefficient depends entirely on the thermodynamic term in equation (3). The effect of the electrophoretic terms is never greater than 0.5 per cent of the diffusion coefficient at concentrations of 0.01 molar or less.

The Conductometric Diffusion Measurement.[1, 2]—The simplest form of cell for an electrolyte diffusing vertically upward is a rectangular parallelepiped of height a, with electrodes at positions which may be most suitably determined from theoretical considerations. The schematic cross-section of such a cell is shown in Figure 1, in which the electrodes are fixed at identical distances ξ from the top and bottom of the cell. Steady-state diffusion occurs in the x-direction. By a suitable technique, affording the minimum of convection, the diffusion process is started in the cell, and at suitable time intervals the conductance is measured at the top and bottom pairs of electrodes. In this method we rely on measurements toward the latter stages of diffusion, when the concentrations at the top and bottom pairs of electrodes differ so little that it is safe to assume constancy of the diffusion coefficient over this small concentration difference. Under these conditions, Fick's second law for restricted diffusion becomes

$$\frac{\partial c}{\partial t} = \mathfrak{D}\frac{\partial^2 c}{\partial x^2}, \tag{6}$$

with the boundary condition

$$\frac{\partial c}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a. \tag{7}$$

The solution of this equation which satisfies these conditions is

$$c = \sum_{n=1}^{\infty} A_n e^{-(n^2\pi^2\mathfrak{D}/a^2)t} \cos\frac{n\pi x}{a} + c_0, \tag{8}$$

where A_n are the Fourier coefficients.

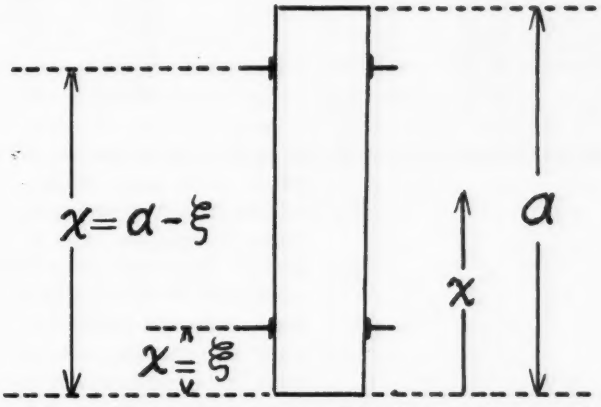


FIG. 1.—Verticle cross-section of a cell showing quantities involved in the conductometric method for determining diffusion coefficients.

The method of applying this equation has great simplicity, for, by subtracting the concentration of electrolyte at the top from that at the bottom, all the even terms of the series vanish. Further, if the electrodes are placed at a distance $\xi = a/6$ from the top and bottom, the third term of the series vanished. As a result, we obtain for the difference of bottom and top concentrations

$$c(\xi) - c(a - \xi) = 2A_1 e^{-(\pi^2\mathfrak{D}/a^2)t} \cos\frac{\pi\xi}{a} + 2A_5 e^{-(25\pi^2\mathfrak{D}/a^2)t} \cos\frac{5\pi\xi}{a}, \tag{9}$$

and only the first term on the right has any significance.

Further, in dilute solutions, it is safe to replace the difference in concentrations at bottom and top by the difference in conductances $(K_B - K_T)$, so that the first-order equation

$$\ln (K_B - K_T) = -\frac{\pi^2\mathfrak{D}}{a^2} t + \text{constant} \tag{10}$$

is readily obtained, since A_1, A_5, ξ, and a are constant. The great simplicity of the method is at once apparent, since a measurement of the depth of the cell, a, and the

top and bottom conductances at suitable time intervals are the only data required to determine the diffusion coefficient. Finally, since we rely on results obtained when the concentrations do not differ greatly, we obtain the "differential" diffusion coefficient as required by theory.

Evidence for the Presence of the Electrophoretic Terms in Equation (4). In Figure 2 values of $(\mathfrak{D}_0 - \mathfrak{D})$, calculated by equations (3), (4), and (5), for the diffusion of lithium[9] and potassium chlorides[2, 10] are plotted against the square root of the molar concentration of the electrolyte. The data required for this computation are compiled in Table 1. The middle curve in Figure 2 represents the complete calculation for potassium chloride solutions, whereas the top curve, denoted $(\overline{\mathfrak{M}}/c)$, is the graph of $(\mathfrak{D}_0 - \mathfrak{D})$ for this salt upon neglecting the electrophoretic terms of equation (4). The difference between these two curves represents the magnitude of the electrophoretic influence. Since the mobilities of the potassium and chloride ions are nearly the same, the second term on the right-hand side of equation (4) is negligible, and only the last term of this equation contributes to the calculation. At 0.01 molar the electrophoretic correction for potassium chloride is 0.5 per cent of the magnitude of its diffusion coefficient. For solutions of lithium chloride, the difference in mobilities of the lithium and chloride ions is such as to cause cancellation of the negative and positive electrophoretic terms of equation (4), so that the complete theoretical result is the same as that obtained by keeping $(\overline{\mathfrak{M}}/c)$ constant and equal to the first term on the right-hand side of this equation.

TABLE 1

QUANTITIES EMPLOYED IN THEORETICAL COMPUTATIONS

	λ°				a (A)
Li^+	38.69	η_0	$= 8.949 \times 10^{-3}$	LiCl	4.0
Na^+	50.11	D	= 78.54	NaCl	4.0
K^+	73.52	T	= 298.16	KCl	3.5
Rb^+	78.05	$\lvert z_1 \rvert$	= 1	RbCl	3.5
Cs^+	77.27	$\lvert z_2 \rvert$	= 1	CsCl	3.5
Ag^+	61.92	ν_1	= 1	$AgNO_3$	3.5
Cl^-	76.34	ν_2	= 1	KNO_3	3.5
NO_3^-	71.44	ν	= 2		

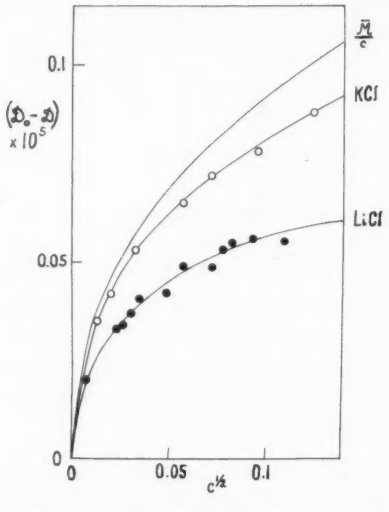


FIG. 2.—Comparison of observed and calculated diffusion coefficients of lithium and potassium chloride solutions. The two lower curves represent the complete calculations according to equations (3) and (4). The top curve, denoted $(\overline{\mathfrak{M}}/c)$, represents the calculation without the electrophoretic terms, the last two terms of equation (4). The diameters of the circles equal 0.1 per cent of the diffusion coefficient.

It is apparent from Figure 2 that excellent agreement of the experimental results with those computed theoretically is obtained. In this figure the diameter of the circles equals 2 in the third decimal place, which corresponds to 0.1 and 0.15 per cent of the diffusion coefficients of potas-

sium and lithium chlorides, respectively. Consequently, these results confirm the validity of the thermodynamic term in equation (3), as well as the magnitude and sign of the electrophoretic effects. It is important to note that above 0.01 molar concentration, deviations from the theoretical prediction occur.[9]

Activity Coefficients from Diffusion Data.—In all our previous calculations we computed the thermodynamic term from the best available activity-coefficient data. Since no accurate experimental determinations of these quantities at concentrations below 0.005 molar were available, it was necessary to rely on extrapolations by the Debye and Hückel equations, the parameters of which had to be determined from data at higher concentrations. We shall now reverse this procedure and, by assuming the validity of equations (3) and (4) at concentrations from 0 to 0.01 molar, compute the activity coefficients from the diffusion-coefficient data. For the calculation of the $(\overline{\mathfrak{M}}/c)$ term, 4.0 A is chosen for the distance parameter, a, for the hydrated electrolytes, lithium and sodium chlorides, and 3.5 A for the other, less hydrated salts.

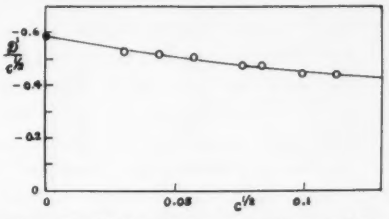


FIG. 3.—Plot used for computing activity coefficients according to equation (13).

Rearrangement of equation (3) yields

$$\frac{\mathfrak{D}}{\nu\ 1000\ RT(\overline{\mathfrak{M}}/c)} - 1 \equiv \mathfrak{D}' = c\,\frac{\partial \ln y_{\pm}}{\partial c}, \tag{11}$$

whence

$$\ln y_{\pm} = \int_0^c \frac{\mathfrak{D}'}{c}\,dc = \int_0^c \frac{2\mathfrak{D}'}{c^{1/2}}\,dc^{1/2}, \tag{12}$$

and

$$\log y_{\pm} = \frac{\ln y_{\pm}}{2.303} = 0.8686 \int_0^c \frac{\mathfrak{D}'}{c^{1/2}}\,dc^{1/2}. \tag{13}$$

At 25 deg.

$$\lim_{c\to 0}\left[\frac{\mathfrak{D}'}{c^{1/2}}\right] = \frac{\partial \ln y_{\pm}}{2\partial c^{1/2}} = -\frac{2.303}{2}\,S_{(f)} = -0.5861, \tag{14}$$

since the limiting theoretical slope of the Debye and Hückel theory, $S_{(f)}$, is 0.5091 at this temperature. By plotting $\mathfrak{D}'/c^{1/2}$ versus $c^{1/2}$, evaluation of log $y_{\pm}$ at suitable concentrations may be easily achieved by graphical integration. Such a graph for potassium chloride solutions is shown in Figure 3. The method is a very sensitive one, since it involves the whole area under the graph. The results of this calculation are recorded in Table 2. The last column contains the bibliographic reference number. Although these results have little significance beyond the third decimal place, we have expressed those for sodium and potassium chloride solutions to the fourth place, in order to compare them with the recent evaluation of this quantity from cells with liquid junction by Shedlovsky,[11] denoted in the table by

NaCl(S) and KCl(S). The agreement between the values derived by those entirely different methods is very encouraging.

TABLE 2

ACTIVITY COEFFICIENTS, $y_{\pm}$, OF THE ALKALI METAL CHLORIDES AND SILVER AND POTASSIUM NITRATES AT 25° FROM DIFFUSION DATA

	CONCENTRATION OF ELECTROLYTE					
	0.0005	0.001	0.002	0.005	0.01	REF.
LiCl	0.975	0.966	0.952	0.929	0.905	9
NaCl	.975	.9653	.9516	.9279	.9027	9
NaCl(S)	...	.9651	.9519	.9273	.9022	11
KCl	.975	.9651	.9517	.9280	.9016	2
KCl(S)	...	.9650	.9516	.9270	.9015	11
RbCl	.975	.966	.951	.928	.901	12
CsCl	.975	.965	.951	.927	.899	13
KNO_3	.975	.964	.951	.926	.901	14
$AgNO_3$	0.974_5	0.963	0.949	0.922	0.894	15

For purposes of future comparison and criticism, the diffusion coefficients of these uniunivalent electrolytes obtained at round concentrations from graphs of our conductometric results are recorded in Table 3. The values at zero concentration were computed by equation (5) from the data in Table 1.

TABLE 3

DIFFUSION COEFFICIENTS OF ALKALI METAL CHLORIDES AND POTASSIUM AND SILVER NITRATES AT ROUND CONCENTRATIONS AT 25° ($\mathfrak{D} \times 10^5$ [CM.^{2} SEC.$^{-1}$])

v	LiCl[8]	NaCl[8]	KCl[9]	RbCl[12]	CsCl[13]	KNO_3[14]	$AgNO_3$[15]
0.000	(1.368)	(1.612)	(1.996)	(2.057)	(2.046)	(1.931)	(1.767)
.001	1.345	1.586	1.964	2.024	2.013	1.899	1.738
.002	1.337	1.576	1.952	2.012	2.001	1.887	1.728
.003	1.331	1.570	1.944	2.003	1.992	1.879	1.720
.005	1.323	1.561	1.933	1.991	1.978	1.866	1.708
.007	1.318	1.554	1.923	1.983	1.969	1.856	1.699
0.01	1.312	1.545	1.915	1.972	1.958	1.844	1.688

* Contribution No. 1140 from the Department of Chemistry, Yale University. This investigation was supported in part by the Atomic Energy Commission under Contract At-(30-1)-1375.

[1] H. S. Harned, and D. M. French, *Ann. N.Y. Acad. Sci.*, **46**, 267, 1945.

[2] H. S. Harned, and R. L. Nuttall, *J. Am. Chem. Soc.*, **69**, 736, 1947.

[3] L. Onsager, and R. M. Fuoss, *J. Phys. Chem.*, **36**, 2689, 1932.

[4] L. Onsager, *Phys. Rev.*, **37**, 405, 1931; **38**, 2265, 1931.

[5] H. S. Harned, *Chem. Rev.*, **40**, 461, 1947; H. S. Harned and B. B. Owen, *The Physical Chemistry of Electrolytic Solutions* (2d ed.; New York; Reinhold Publishing Corp., 1950), pp. 86–90, 129, 178–182.

[6] P. Debye and E. Hückel, *Physik. Z.*, **24**, 185, 1923.

[7] Harned and Owen, *op. cit.*, p. 129.

[8] W. Nernst, *Z. physik. Chem.*, **2**, 613, 1888.

[9] H. S. Harned and C. L. Hildreth, Jr., *J. Am. Chem. Soc.*, **73**, 650, 1951.

[10] H. S. Harned and R. L. Nuttall, *J. Am. Chem. Soc.*, **71**, 1460, 1949.

[11] T. Shedlovsky, *J. Am. Chem. Soc.*, **72**, 3680, 1950.

[12] H. S. Harned and M. C. Blander, *J. Am. Chem. Soc.*, **75**, 2853, 1953.

[13] H. S. Harned and M. C. Blander, *ibid.* (in press).

[14] H. S. Harned and R. M. Hudson, *J. Am. Chem. Soc.*, **73**, 652, 1951.

[15] H. S. Harned and C. L. Hildreth, Jr., *J. Am. Chem. Soc.*, **73**, 3292, 1951.

THE SHOCK TUBE IN AERODYNAMIC AND STRUCTURAL RESEARCH

H. G. STEVER AND R. L. BISPLINGHOFF*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Communicated by E. R. Gilliland; read before the Academy November 11, 1953

A shock tube usually consists of a cylindrical tube of constant cross-section which is divided into two chambers by a frangible diaphragm stretched across the tube. The two chambers initially have different gas pressures, thus putting a differential pressure across the diaphragm. In operation the stressed diaphragm is punctured by a knife or is spontaneously ruptured by increasing the differential pressure beyond the burst point. Following rupture of the diaphragm, pressure waves and expansion waves propagate into the low- and high-pressure chambers, respectively, and in turn are reflected from the open or closed ends of their respective chambers. The irregular pattern of compression waves proceeding into the low-pressure chamber tends to coalesce, forming a normal shock wave which propagates into the low-pressure gas at rest, setting the gas in motion behind the shock wave and also raising the pressure, temperature, and density. The pattern of rarefaction waves propagating into the high-pressure chamber tends to spread out.

Shock tubes with widely differing physical characteristics of length, cross-section, position of diaphragm, pressures, and gases have been used for many purposes. They were initially used to study shock-wave formation and propagation and there still remains considerable work to be done in this field—in the study of shock-wave and expansion reflections, and refraction at gaseous interfaces and at solid surfaces. Related to these are the studies of blast propagation over obstacle shapes of interest. Gaseous detonation and deflagration studies have been made in shock tubes. Shock tubes designed to serve as short-duration wind tunnels, subsonic, transonic, and supersonic, are in use. There are many other special-purpose uses of shock tubes.

The Massachusetts Institute of Technology shock tube has been designed for use in the study of the effect of unsteady pressure forces on aircraft and aircraft structures, as well as for use as a short-duration wind tunnel for the subsonic, transonic, and supersonic regions. For many of the principal features of design, the ideal theory of shock-tube operation can be employed.

One-dimensional compressible-fluid-flow equations suffice to solve the problem. The physical principles of conservation of mass, momentum, and energy and the first and second laws of thermodynamics are required. The statement of these for application to this one-dimensional problem are as follows:[1]

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho\, dx = 0 \qquad \text{(Conservation of mass)}, \tag{1}$$

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho u\, dx = p(x_1, t) - p(x_2, t) \qquad \text{(Conservation of momentum)}, \tag{2}$$

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho\left[\frac{u^2}{2} + e\right] dx = p(x_1, t)u(x_1, t) - p(x_2, t)u(x_2, t) \qquad \text{(Conservation of energy and first law of thermodynamics)}, \tag{3}$$

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \rho s \, dx \geqq 0 \qquad \text{(Second law of thermodynamics).} \tag{4}$$

In these expressions, p, ρ, e, s, and u are, respectively, the gaseous pressure, density, specific internal energy, specific entropy, and particle velocity; x and t are the linear and temporal co-ordinates. The limits of integration are any two x-co-ordinates fixed to the gas, that is, traveling with the gas elements as a function of time.

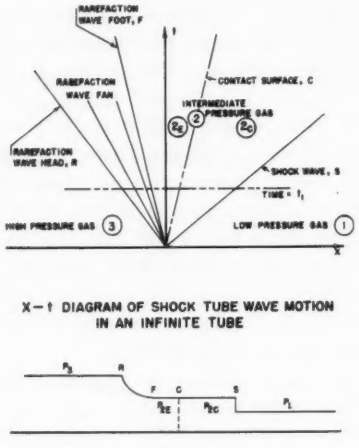


FIGURE 1

The solution of one-dimensional compressible flow by these basic laws has often been carried out,[1] and the resulting application to the ideal shock tube has been made.[1, 2, 3] The results can best be shown on the x-t plot (Fig. 1), showing the wave pattern and gas motion in a doubly infinite shock tube. A shock wave propagates into region *1*, the undisturbed low-pressure gas, with a velocity greater than that of sound. Behind the shock wave, in region *2C*, the intermediate pressure gas is in uniform motion to the right. Its pressure, density, and temperature have been increased. The Rankine-Hugoniot[4] relationship between the shock velocity, U_s, and the particle velocity, u_2, in region *2*, and the pressure ratio across the shock, $_1/p_2$, is as follows:

$$\frac{U_s}{U_s - u_2} = \frac{(\gamma + 1)/(\gamma - 1) + p_1/p_2}{1 + [(\gamma + 1)/(\gamma - 1)]\,(p_1/p_2)}, \tag{5}$$

where γ is the ratio of specific heats. Other relationships required to determine the flow conditions in region *2C* are[2, 3]

$$M_{2C} = \frac{u_{2C}}{a_{2C}} = \frac{[2/(\gamma - 1)]\,(1 - p_1/p_2)}{\sqrt{[2\gamma/(\gamma - 1)]\,\{1 + [(\gamma + 1)/(\gamma - 1)]\,(p_1/p_2)\}}} \tag{6}$$

where M_{2C} is the Mach number of the flow and a_{2C} is the velocity of sound in region *2C* immediately following the shock, and

$$\frac{p_1}{p_3} = \frac{p_1}{p_2}\left\{1 - \frac{1 - p_1/p_2}{\sqrt{[2\gamma/(\gamma - 1)]\,(p_1/p_2)\,[(\gamma + 1)/(\gamma - 1) + p_1/p_2]}}\right\}^{2\gamma\,(\gamma - 1)}. \tag{7}$$

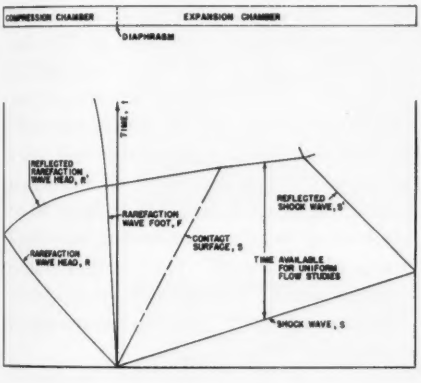
Here p_3 is the initial pressure on the high-pressure side of the diaphragm.

The employment of the ideal theory in the design of a shock tube for aerodynamic and structural research is fairly direct. In such a shock tube there is a test section for aircraft and structural models placed in the low-pressure chamber at a length L_T from the diaphragm. The length of the high-pressure chamber is L_3, and of the low-pressure chamber, L_1. It remains, then, to employ design criteria for these lengths.

The shock wave and the gas flow behind it subject a model in the test section

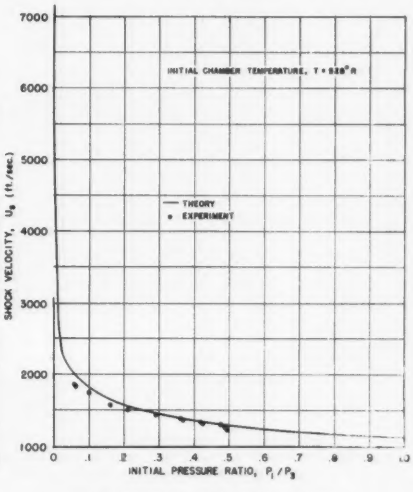
not only to the pressure increase across the shock, but also to the gust, or the gas velocity, following the shock. Thus, for experiments on structural loading, one can employ the shock and the flow in region *2C* behind the shock. For employment as a short-duration wind tunnel, the uniform flow in region *2C* is used. In this application one must allow reflected shocks and the resulting unsteady flow at the model position to die out.

In a shock tube of finite length, the uniform flow in region *2C* is interrupted by the return of the reflected shock from the closed end of the low-pressure chamber, by the arrival of the rarefaction wave reflected from the end of the high-pressure chamber, or by the arrival of the contact surface, as is seen in Figure 2. The time of the arrival of the reflected shock clearly depends on the shock-propagation velocity and the length from the test section to the end of the low-pressure chamber,



X-T DIAGRAM OF SHOCK TUBE WAVE MOTION IN A FINITE TUBE

FIGURE 2



COMPARISON OF THEORY WITH EXPERIMENT FOR THE VARIATION OF SHOCK VELOCITY U_s WITH INITIAL PRESSURE RATIO P_1/P_3

FIGURE 3

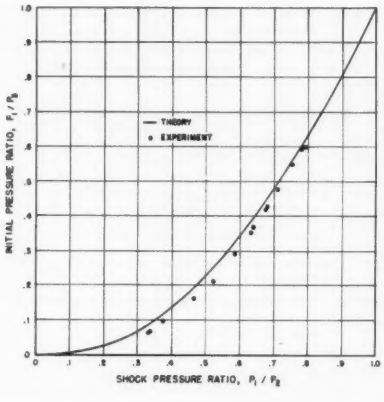
I_S. Actually, the reflected shock returning from the closed end is stronger and has a greater shock-propagation velocity relative to the gas. Since, however, the gas in region *2C* into which it is propagating is in motion to the right, the actual velocity relative to the tube is less than the velocity of the initial shock.

Before using the ideal theoretical shock velocity in design, it is well to see how experiment and ideal theory agree. Figure 3 is a plot of the shock velocity as measured by electronic timing gear, compared with the ideal theoretical shock velocity, both plotted against initial pressure ratio. Figure 4 is a plot of the initial pressure ratio versus the ideal and measured shock pressure ratio. It is seen that the measured values fall somewhat below ideal. This is probably due to the fact that considerable energy is expended accelerating fragments of diaphragm as well as generating turbulent motion as the diaphragm bursts. However, for design the ideal theory is reasonably good.

Figure 5 shows a schematic diagram of the M.I.T. shock tube. It is approximately 98 feet long. The length of the high-pressure chamber is 30.5 feet; from diaphragm to test section is 41 feet; from test section to the end of the low-pressure chamber is 26.5 feet. There are two schlieren optical systems for detection of the passage of the shock wave at stations 3 feet apart immediately upstream of the test section. Electronic timing of the interval between arrival of the shock at each of these stations permits measurement of the shock velocity, U. From the following Rankine-Hugoniot relations,[2] the strength of the shock can then be deduced:

$$U_s = a_1\sqrt{\frac{(\gamma + 1)/(\gamma - 1) + p_1/p_2}{[2\gamma/(\gamma - 1)]\,(p_1/p_2)}} \tag{8}$$

Figure 6 is a picture of this tube.



THEORETICAL AND EXPERIMENTAL CURVES OF SHOCK PRESSURE RATIO P_1/P_2 AS A FUNCTION OF INITIAL PRESSURE RATIO P_1/P_2

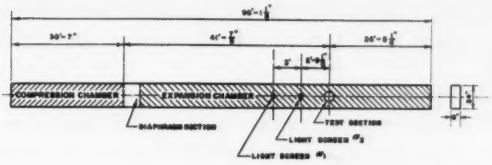
FIGURE 4

Figure 7 is a closeup of the diaphragm section. There are positions for five diaphragms and provision for stepping up the pressure to the full pressure difference in five steps. Thus, for a total pressure difference, across the section of 250 lb., each diaphragm would have only 50 lb. across it. This design was chosen so that the plastic diaphragm material commonly available and having good bursting characteristics could be used. Thicknesses greater than that required to hold 50 or 60 lb. were so great that incomplete shattering often resulted. The first diaphragm is punctured by a knife blade concealed in its center support; the other diaphragms burst from overloading.

Figure 8 shows the interferometric measuring system and its suspension. This interferometer has 7-inch-diameter optics. Flash interferometric pictures of the instantaneous flow over a model, when compared to no-flow pictures, constitute a measurement record of the density throughout the flow field. From this density field, the velocity and pressure fields can be calculated, thus yielding a measure of the pressures and forces on models.

The qualities of the shock tube as a short-duration wind tunnel can now be investigated. In this use the uniform flow in region *2C* is employed. In theory, for air, Mach numbers up to a limiting 1.89 can be obtained. However, the density of the air is so low and the time duration of the flow is so short for the very high range of Mach number that the practicably usable Mach number range ends somewhat lower.

For use as a wind tunnel the transient flow over a model due to the buildup of circulation and the reflected shock waves must be allowed to become small. For the reflected-shock-wave effects, a few chord lengths of flow are sufficient to permit them to decrease to small values. In subsonic flow the buildup of circulation is substantially completed, say, within 95 per cent, in 10 model chord lengths of



APPROXIMATE INSIDE DIMENSIONS OF THE SHOCK TUBE AT M.I.T.

FIGURE 5

FIGURE 6

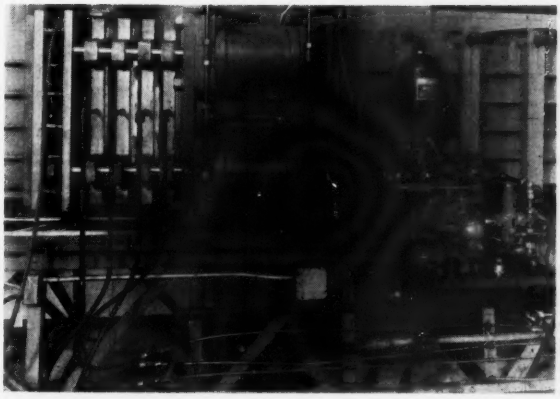


FIGURE 7

flow. In supersonic flow, pressure forces on models are established more quickly. In some work it was desirable to have 10 chord lengths of flow over our standard aircraft wing model of 4-inch chord, at a low velocity of flow of about 100 ft/sec. Using this as a design point, the resulting flow duration as a function of Mach number and flow velocity is shown in Figure 9. In this figure it is shown that the

duration of steady flow is limited by the arrival of the reflected shock wave for the low Mach number range, from 0 to about 0.65, and by the arrival of the contact surface above 0.65.

The Reynolds number is an important parameter in aerodynamic measurements. Figure 10 is a plot of the Reynolds number attainable as a function of Mach number and flow velocity. This plot is for a 4-inch-chord model. It is seen that the Reynolds number is one million or greater for flows over a Mach number range from 0.05 to 1.2.

FIGURE 8

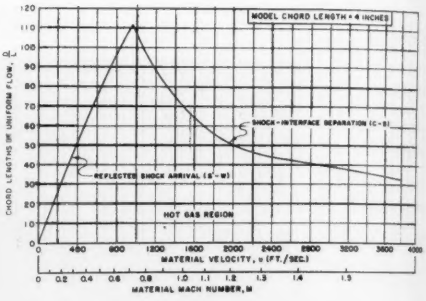

MODEL CHORD LENGTH = 4 INCHES
SHOCK - INTERFACE SEPARATION (C - S)
REFLECTED SHOCK ARRIVAL (S - W)
HOT GAS REGION
MATERIAL VELOCITY, u (FT./SEC.)
MATERIAL MACH NUMBER, M

FIGURE 9

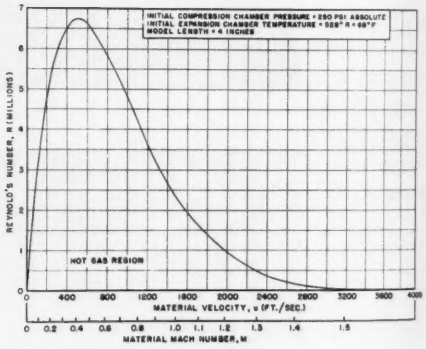

INITIAL COMPRESSION CHAMBER PRESSURE = 250 PSI ABSOLUTE
INITIAL EXPANSION CHAMBER TEMPERATURE = 528° R = 68° F
MODEL LENGTH = 4 INCHES
HOT GAS REGION
MATERIAL VELOCITY, u (FT./SEC.)
MATERIAL MACH NUMBER, M

FIGURE 10

The results shown in Figures 9 and 10 illustrate the fact that flows useful for aerodynamic measurements are obtainable from Mach numbers slightly above 1 to Mach numbers well into the subsonic region, and extend through the transonic region. These regions have been explored by several experimenters. In one of the first works discussing the shock tube as a short-duration wind tunnel, Geiger and Mautz[2] studied low supersonic flow over wedge-shaped airfoils. These studies included experiments on attached and detached shocks. They also studied mixed flow over double-wedge airfoils at high subsonic speeds.

The primary application of a shock tube for structural studies is in the development of methods for predicting damage to structures from gusts. It may be said that this application has two aspects. The first is the experimental determination of the pressure distribution and total forces on rigid model structures,

and the second is the measurement of the response of deformable structures to these pressures and forces.

The discussion which follows considers these problems separately. Since one of the principal reasons for development of the shock tube described by the present paper was for aircraft structural investigations, the applications considered here are for the most part limited to aircraft structures.

1. *Experimental Determination of the Pressures and Forces on Model Structures.*—The design of structural components to withstand envelopment by strong shock waves must be based upon accurate data on pressure distributions and forces. Although theory can be used to predict this information in certain simple cases, the structures of interest are usually much too complex for theoretical analyses. As a result, the structural engineer must rely largely on experimental data for design loadings. Complete loading information on structural components includes the time history of the pressure distribution as well as the time history of the total loads.

Shock tubes of the type described here are ideally suited for measurement of pressures and forces on structural components when the component is essentially two-dimensional in character. Overpressures or diffraction pressures can be obtained from interferometer pictures taken at successive instants throughout the development of the flow pattern. This must be done by successive experiments involving a single interferometer picture in each experiment or by a succession of pictures in a single experiment. Obviously the success of the former technique, one which has been employed extensively in the past, requires faithful duplication of test conditions in the tube during each experiment. Pressure gauges with very high-frequency response characteristics mounted flush on the model surface can also be used to obtain diffraction loading data. Their output can be recorded by means of an oscilloscope and a high-speed motion-picture camera.

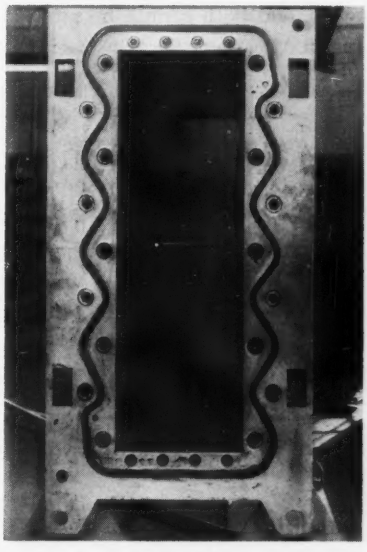
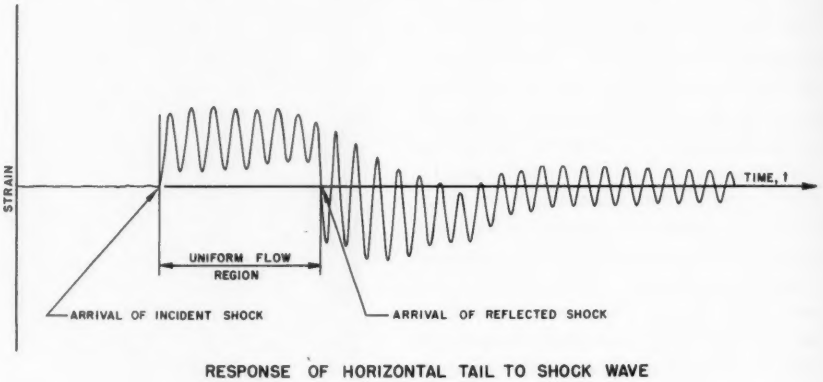


FIGURE 11

The measurement of total forces and moments on a model during the gust phase can be accomplished by mounting the model on a wind-tunnel-type balance system. Because of the suddenness in load application, the frequency response characteristics of the balance system must be carefully considered in its design.

2. *Experimental Determination of the Response of Structures.*—In general, it can be said that shock tubes with cross-sections of the size described here cannot be used as a means of determining ultimate failing *strengths* of given full-size aircraft structures by testing structural models. The principal reason for this is that it is not feasible to reproduce faithful structural scale models if the scale ratio is much

less then 1–4. For most aircraft structural components, this scale ratio would require a tube of the order of 10–15 feet in diameter. However, if a mathematical model of the behavior of a given structure can be constructed, a very small model with scale ratios less than 1–4 can be tested to give useful results. A simple example of the kind involves structural behavior in the elastic range. Since the aircraft structural engineer can tolerate little or no permanent set in the working stress range, he is often interested in stresses below the yield point. It is usually feasible to construct small models which reproduce, with a sufficient degree of precision, the *elastic* and inertial properties of a given aircraft structure. Such models need not duplicate the construction details of the full-scale structure, and, in fact, their load-carrying system is often entirely different. It is worth noting here that gravity forces have a negligible effect on the response of aircraft structural components, whereas they are sometimes very important in the case of large civil engineering structures. This fact increases the complexity of structural modeling of buildings, since the requirements on inertial scaling and gravity scaling are different, and both cannot be satisfied simultaneously.



RESPONSE OF HORIZONTAL TAIL TO SHOCK WAVE

FIGURE 12

An important engineering use of a shock tube in structural-response studies is in performing the function of an analogue computer when the mathematical model of a given structure is known and the aerodynamic pressure distribution is not known. An important illustration of this use is that of measuring the dynamic response of aircraft horizontal tail structures. In this problem it is possible to construct a mathematical model *elastically* and inertially similar to a given full-scale structure. However, the distribution and time history of the loading are difficult to predict. Thus the shock tube can serve the purpose of an analogue computer by supplying the physically correct force inputs to a mathematical structural model of the tail. Figure 11 illustrates a photograph of a model of a hypothetical tail structure mounted in the test section of the M.I.T. shock tube at a $12^1/_4°$ angle of attack. The model illustrated here is a rectangular cantilever model with an NACA 0009 airfoil section. The response of the model was measured by electric strain gauges mounted on opposite sides of the tail spar near the root. A typical strain-time

record is shown by Figure 12. When the model is struck by the shock, it begins to vibrate and tends to seek an equilibrium strained position corresponding to the steady-state air forces produced on the model in the "uniform" material-velocity field behind the shock. At approximately 0.043 second after the arrival of the incident shock, the model is struck from the rear by the shock which has been reflected from the expansion-chamber end-wall of the tube. Since there is zero material velocity in the region behind this reflected shock, the model is then in a still field and thus tends to vibrate about its initial unstrained position. Subsequently, the model is observed to respond to the various waves which continue to reflect from the tube ends. In this record, the features of most practical interest are the peak strain and the strained position about which vibrations take place during the passage of the uniform material-velocity field immediately behind the incident shock.

* The detailed design and early construction of the M.I.T. shock tube was the responsibility of Mr. Thomas Parsons, Mr. Edward Godfrey, and Mr. Kenneth Rathbun. The final assembly, testing, and operation has been under the direction of Dr. Emmett Witmer, assisted by Mr. Vaughn Beals. The latter two have carried out the structural experiments described herein. The shock tube is the property of the United States Air Force and all design and constuction work was sponsored under Air Force contract.

[1] R. Courant and K. O. Friedricks, *Supersonic Flow and Shock Waves* (New York: Interscience Publishers, Inc., 1948).

[2] F. W. Geiger and C. W. Mautz, *The Shock Tube as an Instrument for the Investigation of Transonic and Supersonic Flow* (University of Michigan Engineering Research Institute, United States Navy Contract No. N6-ONR-232, June, 1949).

[3] J. Lukasiewicz, *Shock Tube Theory and Applications* ("National Aeronautical Establishment Reports," No. 15 [Ottawa, Ontario, Canada, 1952]).

[4] W. J. M. Rankine, "On the Thermodynamic Theory of Waves of Finite Longitudinal Disturbance," *Trans. R. Soc. London* **160**, 277–288, 1870; H. Hugoniot, "Sur la propagation du mouvement dans les corps et specialement dans les gaz parfaits," *Journal de l'ecole polytechnique*, **58**, 1–125, 1889.

GRID ROTATION IN LÜDERS BANDS

BY T. Y. THOMAS

NAVAL RESEARCH LABORATORY, WASHINGTON, D.C.

Communicated March 25, 1954

1. *Preliminary Remark.*—In this note, one of the mathematical conditions imposed in our three previous notes in these PROCEEDINGS[1] has been modified in order to secure a proper agreement between the calculated and experimentally determined values of the rotation of the grid lines within the Lüder, or plastic, band. In the following discussion these notes will be referred to by the numerals "I" "II," and "III," according to the order of their appearance, and references to equations in them will be made by placing the appropriate numeral before the equation in question.

We shall confine our attention to the case of incompressible plastic material under the von Mises yield condition. For this case the basic solution is given by equation I(11), or

$$u_1 = ay_2;\ u_2 = by_2;\ u_3 = -by_3, \tag{1}$$

relative to the y co-ordinate system.

2. *The Plastic Contraction.*—The above quantity b in equation (1) represents contraction per unit length in the thickness of the plastic band and is the analogue of the elastic quantity $\nu\beta$ in the third equation III(1). Now in the elastic case this contraction is determined for materials of known elastic moduli when the tension is assigned. In the plastic case the equations at our disposal do not permit a corresponding determination of the contraction b. Indeed, such a determination should not be possible for the plastic state, since the contraction b is influenced in practice by certain conditions which underlie the test and which would appear to be entirely extraneous from the standpoint of the classical elasticity theory. For example, it is known that the depth of the plastic groove, which is measured by b, will be increased when the length of the bar is increased, and in accordance with this fact it is supposed that a certain amount of the elastic energy in the bar is transmitted to the plastic band in the process of its formation. Unless this and possibly other relevant factors are to be made an inherent part of the theory, we must therefore determine the plastic contraction b by actual measurement. We shall adopt this latter procedure in the present discussion. This amounts to assigning to b the value of the experimental ratio $\Delta T/T$ in Table 1 in Note III.

The above choice of the quantity b will affect in no way the formal discussion of the incompressible case subject to the von Mises yield with which we are here concerned. In fact, we see from page 263 of Note I that only the ratio $a/b = \pm\sqrt{8}$ of the constants a and b is determined, and hence we have, as before, $\tan\theta = \pm 1/\sqrt{2}$ from equation I(18). The usual values of $\theta = \pm 35°16'$ are thus retained. Owing to the fact that the experimental value of b is considerably larger than the value of this quantity originally selected in Note I, the contraction within the plastic band, perpendicular to the flat side of the bar, will exceed the corresponding elastic contraction. Consequently, the plastic region will be drawn away from the elastic region along small strips in the planes of separation of these regions, as indicated by lines such as AB in Figure 1. However, the proper boundary conditions will continue to be satisfied over the flat surfaces of the plastic band, which are represented in Figure 1 by the lines BC and DE. This separation of material over the boundaries of the plastic band is the price we have to pay for the above choice of the quantity b and the simplicity of solution (1) under consideration. In this connection it may be mentioned that the modification proposed in section 5 would appear to be more acceptable from the physical standpoint and may be found to have some basis in fact on close examination of the character of the surface of the plastic band.[2]

3. *Rotation Formulae.*—Choosing the + sign in the expression for $\tan\theta$ in section 2, so that $\tan\theta = 1/\sqrt{2}$, we have, correspondingly, $a = 2\sqrt{2}b$. Using these relations and also the fact that $\sin\theta = 1/\sqrt{3}$ and $\cos\theta = \sqrt{2}/\sqrt{3}$, we now have

$$\sin(\theta - \omega) = \frac{\sqrt{2}}{3}(1 + \nu)\beta;\ \cos(\theta - \omega) = 1$$

from equation III(17). From III(15), we find

$$\sin \omega = \frac{1}{\sqrt{3}}\left[1 - \frac{2}{3}(1 + \nu)\beta\right].$$

Hence equation III(9) becomes

$$m_2 = \frac{-\sqrt{2}\,[b - (1 + \nu)\beta/3]}{1 - b}, \tag{2}$$

where we have neglected terms involving the small quantity βb. Similarly, from III(16) we have

$$\cos \omega = \frac{\sqrt{2}}{\sqrt{3}}\left[1 + \frac{(1 + \nu)\beta}{3}\right],$$

and hence III(10) becomes

$$m_1 = \frac{1 + 2b}{\sqrt{2}\,[b - (1 + \nu)\beta/3]}. \tag{3}$$

Formulae (2) and (3) give the slopes of the original grid lines $x_2 = \text{const.}$ and $x_1 = \text{const.}$, respectively, in the plastic band before unloading. However, it was observed in Note III that the effect of unloading produced a negligible rotation of the grid lines. Hence the final slopes of the grid lines may be considered as given

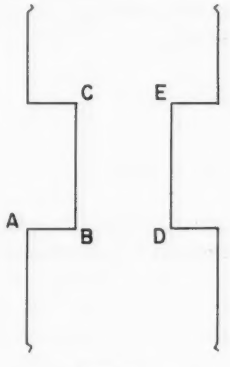


FIGURE 1

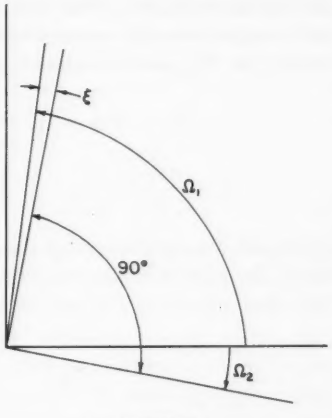


FIGURE 2

by equations (2) and (3). Denote by Ω_1 and Ω_2 the inclinations of the lines having the above slopes m_1 and m_2, respectively. Now $m_2 < 0$ from equation (2), and $m_1 > 0$ from equation (3). Hence $\Omega_1 > 0$ and $\Omega_2 < 0$. Now consider

$$\tan(\Omega_1 - \Omega_2) = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{1 + b}{-3\sqrt{2}b^2}, \tag{4}$$

where we have again neglected the quantity βb. Since the right-hand member of equation (4) is negative, it follows that $\Omega_1 - \Omega_2$ is a second-quadrant angle. Hence we can write

$$\Omega_1 = 90^\circ + \Omega_2 + \xi, \tag{5}$$

with $\xi > 0$. But $90° - \Omega_1$ is the angle through which the vertical grid line ($x_1 =$ const.) is rotated (see Fig. 2). Hence, from equation (5), this angle is given by $-\Omega_2 - \xi$. The vertical grid line is therefore rotated through a smaller angle than the horizontal one ($x_2 =$ const.), the difference between these two angles of rotation being ξ. But from equations (4) and (5) we have

$$\tan \xi = \frac{3\sqrt{2}b^2}{1 + b}.$$

Using the values of b from the table in section 8 of Note III, we find that the values of ξ as determined by the above formula differ from zero by amounts which are well within the experimental error. Hence, to within this error, we have $\xi = 0$, and the grid lines remain perpendicular after plastic rotation.

4. *Band Elongation and Slip.*—The formula for the band elongation B_e in section 5 of Note III becomes

$$B_e = b - \frac{1}{3}(2 - \nu)\beta$$

when we substitute for $\sin \theta$ and neglect squares and higher powers of β. We now have $B_e > 0$ for the experimental values of b under consideration.

Let us now examine the possibility of *slip* along the planes separating the plastic band from the elastic parts of the bar. Using the determination of the trigonometric functions in section 3 and neglecting β^2 and higher powers of β, as well as the small quantity βb, the plastic displacement III(8) becomes

$$\left.\begin{aligned} \bar{x}_1 &= (1 - b)x_1 + \sqrt{2}\left[b - \frac{(1 + \nu)\beta}{3}\right]x_2, \\ \bar{x}_2 &= -\sqrt{2}\left[b - \frac{(1 + \nu)\beta}{3}\right]x_1 + (1 + 2b)x_2. \end{aligned}\right\} \tag{6}$$

We now consider the corresponding elastic displacement in the x_1, x_2-plane. The occurrence of the plastic band produces a displacement, in the direction of the x_2-axis, of the elastic parts of the bar by amounts

$$\pm\left[b - \frac{(2 - \nu)\beta}{3}\right]d \tag{7}$$

beyond that given by the purely elastic displacement III(1), where the plus sign applies to the elastic part of the bar for which $y_2 > 0$, and the minus sign to the part for which $y_2 < 0$. This is easily seen from the discussion in section 5 of Note III. In addition, the dynamical conditions on which our work is based allow the possibility of an unrestricted rigid displacement of the elastic portions along their planes of separation with the plastic band. Such a displacement of magnitude g, parallel to the x_1, x_2-plane, will produce changes in the x_1- and x_2-components of the elastic displacement of amounts $g \cos \theta$ and $g \sin \theta$, or $\sqrt{2}g/\sqrt{3}$ and $g/\sqrt{3}$, respectively. Hence we must consider the possibility of elastic displacements in the x_1, x_2-plane of the form

$$x_1' = (1 - \beta\nu)x_1 + \frac{\sqrt{2}g}{\sqrt{3}}; \quad x_2' = (1 + \beta)x_2 \pm \left[b - \left(\frac{2 - \nu}{3}\right)\beta\right]d + \frac{g}{\sqrt{3}}. \tag{8}$$

With regard to displacements in the x_3- or y_3-direction, these are determined by the quantities $\nu\beta$ and b, in accordance with the third equation III(1) and the third equation I(11) for the elastic and plastic regions, respectively.

We now apply equations (6) and (8) to the problem of determining the slip along the planes of separation of the elastic and plastic parts of the bar. In this consideration, the co-ordinates x_1, x_2 in these equations must lie on the lines III(3) in the x_1, x_2-plane, i.e., the lines

$$x_2 = (\tan \omega)x_1 \pm \frac{d}{1+\beta} = [1 - (1+\nu)\beta]\frac{x_1}{\sqrt{2}} \pm (1-\beta)d, \tag{9}$$

in which the plus and minus signs correspond, respectively, to the plus and minus signs in expression (7) and in the second equation (8). Hence we can eliminate the co-ordinate x_2 from the right-hand members of equations (6) and (8) by the substitution (9). This leads to the following relations:

$$x_1' - \bar{x}_1 = \frac{1}{3}(1-2\nu)\beta x_1 - \sqrt{2}\left\{\pm\left[b - \left(\frac{1+\nu}{3}\right)\beta\right]d - \frac{g}{\sqrt{3}}\right\}, \tag{10}$$

$$x_2' - \bar{x}_2 = \left(\frac{1-2\nu}{3\sqrt{2}}\right)\beta x_1 - \left\{\pm\left[b - \left(\frac{1+\nu}{3}\right)\beta\right]d - \frac{g}{\sqrt{3}}\right\}. \tag{11}$$

For $\nu \neq 1/2$ the above co-ordinate differences can be made arbitrarily large by increasing x_1 sufficiently. *Obviously, this situation does not correspond to the physical facts.* Let us therefore assume $\nu = 1/2$. Then equations (10) and (11) become

$$x_1' - \bar{x}_1 = -\sqrt{2}\left[\pm\left(b - \frac{\beta}{2}\right)d - \frac{g}{\sqrt{3}}\right], \tag{12}$$

$$x_2' - \bar{x}_2 = -\left[\pm\left(b - \frac{\beta}{2}\right)d - \frac{g}{\sqrt{3}}\right]. \tag{13}$$

Defining the slip S parallel to the x_1, x_2-plane as the distance between the points (x_1', x_2') and $(\bar{x}_1, \bar{x}_2)$, it follows from equations (12) and (13) above that S is *constant* and depends on the values of the quantities b, d, and g.

How it may reasonably be supposed that the cohesive forces within the material will tend to cause S to have its minimum value, zero, which is, in fact, a possibility, as can be seen from equations (12) and (13). Hence, making the assumption that $S = 0$, we have

$$g = \pm\sqrt{3}\left(b - \frac{\beta}{2}\right)d. \tag{14}$$

Equations (14) give the values of g for zero slip S in terms of the plastic contraction b and the quantity d, by which the width of the plastic band is determined. The resulting displacement of the bar is illustrated in Figure 3, where the plane of separation AB corresponds to the plus sign in equation (14) and the plane CD to the minus sign in this equation. Bars subjected to tension tests have been observed to have the deformation shown in Figure 3 after the appearance of the plastic band.

5. *A Suggestion by G. R. Irwin.*—It has been suggested by G. R. Irwin that solution (1), in which the constant b represents the total or maximum contraction in the plastic band and which results in a band whose cross-section is shown in

Figure 1, be replaced by a sequence of such solutions, with the values of b differing slightly for successive solutions of the sequence. In this way one can produce a band whose cross-section has the appearance shown in Figure 4. The values of b selected may be assumed to vary through discrete steps from the value $b = \nu\beta$ used in Note I to the value giving the experimental or maximum contraction of the band with which we have been concerned in the foregoing discussion. By taking a sufficiently large selection of values b, we can thus evidently approximate the surfaces of the plastic bands observed experimentally.

For any elementary band in the configuration we shall have $\tan\theta = \pm 1/\sqrt{2}$ and $a = \pm\sqrt{8}b$. The boundary condition will be satisfied on the free surface, parallel to the flat side of the bar, of any elementary band, as we see immediately from the foregoing discussion (see sec. 5 of Note I). The configuration represented by Figure 4 will therefore be possible if we can show that the determination of the stresses in two adjacent elementary bands will be such that the conditions for equilibrium will

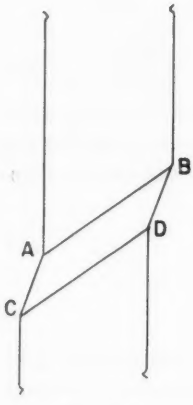


FIGURE 3

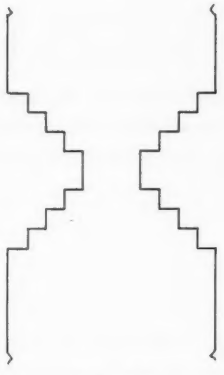


FIGURE 4

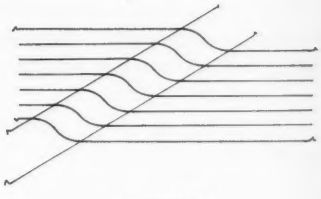


FIGURE 5

be satisfied over their plane of separation. For this purpose we consider equations (6), (12), (13), and (14) of Note I. Combining equations I(6) and I(12), we have $(a^2 + 4b^2)\varphi^2 = 4k^2$. This gives $\varphi = \tau/3b$ on account of the relations $a = \pm\sqrt{8}b$ and $\tau = \sqrt{3}k$. Hence $p = -\tau/3$ from equation I(14). Taking $a = \sqrt{8}b$, equations I(13) now become

$$\sigma_{11} = \frac{\tau}{3};\ \sigma_{12} = \frac{\sqrt{2}\tau}{3};\ \sigma_{13} = 0,$$

$$\sigma_{22} = \frac{2\tau}{3};\ \sigma_{23} = 0;\ \sigma_{33} = 0.$$

It follows that the required equilibrium conditions are satisfied over the planes of separation of adjacent elementary bands, since the components of the stress tensor in these bands are identical. The plastic-band configuration represented by Figure 4 is therefore a possibility.

Figure 5 shows the effect on the horizontal grid lines produced by the type of plastic band under discussion. Near the edges of the band the lines will be shifted only slightly from the horizontal direction, and the maximum rotation will be attained along the center of the band, where the depth of the groove is greatest. The vertical grid lines will of course be distorted into lines which are very approximately perpendicular to the lines in Figure 5 (see sec. 3). Observation shows the shift of the grid lines to be of this nature, rather than as indicated in Figure 1 of Note III.

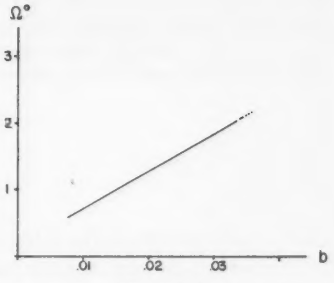


FIGURE 6

Consider the constants a and b for a particular elementary band, and let a' and b' denote the corresponding constants for an adjacent elementary band. For definiteness we suppose $b' > b$. By recourse to the first equation (1), we see immediately that the slip S along the plane of separation of these elementary bands is given by $(a' - a)y_2$, where y_2 denotes the distance of the plane of separation from the center of the plastic band. Putting $\Delta b = b' - b$ and using the relations $a = \sqrt{8}b$ and $a' = \sqrt{8}b'$, it follows that this slip is equal to $\sqrt{8}y_2\Delta b$. By choosing the increments Δb sufficiently small, the slip between adjacent elementary bands can be reduced to molecular dimensions. Slips of such magnitudes along so-called *slip* or *glide planes* have been considered by certain authors in discussing the plastic properties of crystals.[2]

6. *Numerical Results.*—The value of m_2 depends on the yield tension through the term involving β in the right-hand member of equation (2). However, this term is so small in comparison with the values of b under consideration that it can be neglected without serious effect. The formula for m_2 then becomes $m_2 = -\sqrt{2}b/(1 - b)$. Using this expression for m_2, in which b is assumed to be the maximum contraction in the plastic groove, we have computed the rotation Ω of the grid lines,[3] where $\tan \Omega = -m_2$. The values of Ω so determined will give the rotation of the grid lines along the center of the groove on the basis of the discussion in section 5. These computations are entered in Table 1 and are represented graphically in Figure 6.

TABLE 1

b	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Ω (deg.)	0.82	1.65	2.50	3.37	4.26	5.16	6.08	7.01	7.96	8.93

[1] T. Y. Thomas, "On the Inclination of Plastic Slip Bands in Flat Bars in Tension Tests," these PROCEEDINGS, **39**, 257–265, 1953; "The Effect of Compressibility on the Inclination of Plastic Slip Bands in Flat Bars," *ibid.*, pp. 266–273; and "On the Rotation of Grid Lines Produced by the Formation of Plastic Bands in Tension Tests," *ibid.*, **40**, 401–407, 1954.

[2] See, for example, F. Seitz, *The Physics of Metals* (New York: McGraw-Hill Book Co., Inc., 1943), chap. vi.

[3] Cf. G. R. Irwin, "Angle and Strain Relations in Flat Plate Lüders' Bands," *J. Appl. Mech.*, **20**, 449–450, 1953. In eq. (6) in this paper we can identify ϵ with the plastic contraction b, and

the derivative $\partial u'/\partial y'$ with 2Ω. Then, putting $\Omega = \tan \Omega$, as is possible for small angles of rotation Ω, and assuming $\tan \theta = 1/\sqrt{2}$, so that $\tan 2\theta = 4\sqrt{2}$, eq. (6) in Irwin's paper becomes $\tan \Omega = \sqrt{2}b$. But this latter relation also results from our formula for Ω in sec. 6, when we neglect the small quantity b in comparison with 1 in the denominator of the expression for m_2.

A DISCUSSION OF THE LOAD DROP AND RELATED MATTERS ASSOCIATED WITH THE FORMATION OF A LÜDERS BAND

BY T. Y. THOMAS

NAVAL RESEARCH LABORATORY, WASHINGTON, D.C.

Communicated May 19, 1954

1. *Introduction.*—In a fixed-grip experiment resulting in the occurrence of a Lüders band in a flat bar there is a distinct drop in the load at the instant of formation of the band. We have here derived a formula relating this load drop to the width D of the Lüders band and the length L of the bar. The formula involves, in addition, the plastic contraction b introduced in our previous communication "Grid Rotation in Lüders Bands" (these PROCEEDINGS, **40,** 565–572, 1954), on which the following discussion is largely based. Thus we have limited our attention to the use of the von Mises yield condition and have moreover taken Poisson's ratio $\nu = 1/2$, which is equivalent to the assumption that the material is elastically incompressible. This latter assumption is generally admissible, since the elastic displacements are small in comparison with the plastic displacements. In this connection we refer also to the Remark in sec. 2 concerning the value of Poisson's ratio.

References to earlier notes in these PROCEEDINGS as "Notes I, II, and III" will be made as explained in the above-mentioned communication, which will, in turn, be referred to as "Note IV" in the following discussion.

2. *Derivation of the Formula.*—Consider the bar at the yield point just before the formation of the Lüders band. The flat surface of the bar at the instant in question is represented by the rectangle $ABCD$ in Figure 1. Planes P_1' and P_2' bound the elastically deformed material destined to go into the plastic or Lüders band, which is bounded by the planes P_1 and P_2 indicated by the dotted lines. Lengths such as l_1 and D, etc., which will enter the discussion are clearly defined by reference to Figure 1.

Let l denote the unstrained length of the bar which has length $l_1' + l_2'$ at the yield point. Then

$$(l_1' + l_2') - l = l\beta, \tag{1}$$

where the elongation $\beta = \tau/E$, with τ the yield tension and E Young's modulus. After yielding has occurred and the planes P_1' and P_2' go into the positions P_1 and P_2, the unstrained length l will have the strained length $l_1 + l_2$. Hence we can write

$$(l_1 + l_2) - l = l\bar{\beta}, \tag{2}$$

where $\bar{\beta}$ is the elastic elongation after yielding. In relation (2) we have $\bar{\beta} = \bar{\tau}/E$, in which $\bar{\tau}$ is the tension in the bar after the formation of the Lüders band. But $l_1 + l_2 + 2m = l_1' + l_2'$, as we see by reference to Figure 1. Hence, combining relations (1) and (2), we find

$$\frac{l(\tau - \bar{\tau})}{E} = 2m. \tag{3}$$

Now $L = l + d/(1 + \beta)$, where L is the total unstrained length of the bar. But $D' = d \cos\theta$ and $D = (d + 2m) \cos\theta$, where θ is the inclination of the Lüders band. We shall also make use of the band elongation B_e which was defined by the equation $B_e = (D - D')/D'$ in sec. 5 of Note III. Using these relations, we now readily find that equation (3) can be written

$$\frac{D}{(1 + B_e)\cos\theta}\left[B_e + \frac{\tau - \bar{\tau}}{(1 + \beta)E}\right] = L\left(\frac{\tau - \bar{\tau}}{E}\right). \tag{4}$$

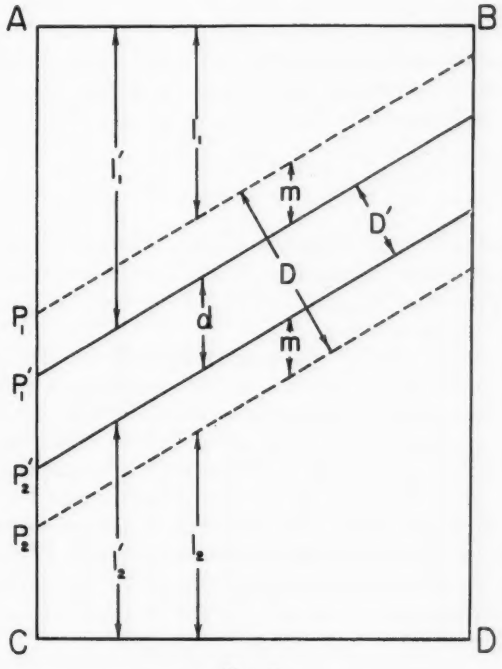


FIG. 1

The second term in the bracketed expression above is small in comparison with the band elongation B_e. Hence, disregarding this term and putting $\cos\theta = \sqrt{2}/\sqrt{3}$,

as found in Note I under the assumption of the von Mises or quadratic yield condition, equation (4) becomes

$$\frac{D}{L} = \frac{\sqrt{2}}{\sqrt{3}} \frac{1 + B_e}{B_e} \left(\frac{\tau - \bar{\tau}}{E} \right). \tag{5}$$

If we take $\nu = 1/2$ in the formula for B_e in sec. 4 of Note IV, we obtain $B_e = b - \beta/2$. Making this substitution in equation (5), we now have

$$\frac{D}{L} = \frac{\sqrt{2}}{\sqrt{3}} \frac{1 + b - \beta/2}{b - \beta/2} \left(\frac{\tau - \bar{\tau}}{E} \right). \tag{6}$$

In this relation $\tau - \bar{\tau}$ is readily determined from the load drop which is observed at the time of formation of the Lüders band. The quantity b is the plastic contraction per unit length in the direction normal to the flat side of the bar and is to be determined by measurement along the central portion of the band, as explained in Note IV. Also, D is the width of the band, as illustrated in Figure 1, and, as previously stated, L is the unstrained length of the bar. Finally, the elongation $\beta = \tau/E$, where τ is the yield tension and E is Young's modulus for the material.

Remark: The value $\nu = 1/2$ has been chosen in deriving relation (6) above because of the situation regarding slip along the planes of separation of the plastic and elastic parts of the bar which we encountered in sec. 4 of Note IV. If we take $\nu = 1/3$, which is the approximate value of Poisson's ratio for the materials commonly used as test specimens in Lüders band experiments, the expression for the band elongation B_e becomes $b - 5\beta/9$. Hence equation (5) gives

$$\frac{D}{L} = \frac{\sqrt{2}}{\sqrt{3}} \frac{1 + b - 5\beta/9}{b - 5\beta/9} \left(\frac{\tau - \bar{\tau}}{E} \right) \tag{7}$$

in place of equation (6). The numerators and denominators of the second ratios in the right-hand members of equations (6) and (7) will thus differ by a term of the amount $\beta/18$. Because of the small values of β involved, such a term will exert no practical influence in the relation.

3. *Meaning of the Width* D.—The actual cross-section of the groove of a Lüders band is of the nature of the curved line ADG in Figure 2, while for the simple theory here under consideration the cross-section of the groove will have the rectangular character indicated by the dotted line $BCEF$. The width D of the Lüders band is represented by the side CE of this rectangle. We are thus confronted with the question as to how D is to be measured in actual practice.

To give a precise answer to the above question, let us consider the plastic contraction b, which we have previously defined by the expression $(T - t)/T$, where T is the thickness of the unstressed bar and t is the contracted thickness of the Lüders band, as shown in Figure 2 of Note III. Hence we can write

$$\frac{Db}{L} = \frac{D(T - t)}{LT} = \frac{2G}{S}, \tag{8}$$

where G is the cross-sectional area of the rectangular groove and S is the area of the side surface of the unstressed bar. Let us now use relation (8) to eliminate D from equation (6). We thus obtain

$$\frac{G}{S} = \frac{(1 + b - \beta/2)b}{\sqrt{6}(b - \beta/2)} \left(\frac{\tau - \bar{\tau}}{E}\right). \tag{9}$$

In relation (9) we identify G with the area of the cross-section of the actual groove represented by the curve ADG in Figure 2. *The width* D *will therefore be such that the area* G *of the rectangle* BCEF *is equal to the area of the cross-section of the actual groove.* This meaning of D is natural, since it would appear that the cross-sectional area of the groove is one of the essential characteristics of the Lüders band.

4. *Approximations to the Formula.*— For plastic displacements which are large in comparison with the elastic displacements, the terms involving β in relation (9) can be neglected, and hence this relation becomes

$$\frac{G}{S} = \frac{1 + b}{\sqrt{6}} \left(\frac{\tau - \bar{\tau}}{E}\right). \tag{10}$$

Relation (10), in turn, can be approximated by

$$\frac{G}{S} = \frac{1}{\sqrt{6}} \left(\frac{\tau - \bar{\tau}}{E}\right), \tag{11}$$

since the plastic contraction b is usually negligible in comparison with unity. Formula (11) appears to be of interest in that it involves only a relation between the load drop, as determined by $\tau - \bar{\tau}$, the area S of the side surface of the bar, and the cross-sectional area G of the groove of the Lüders band.

Remark: An interesting modification of the above formulas is obtained by considering volumes instead of the areas G and S. Thus we easily see that we can write $WG = v \cos\theta$, where W is the width of the bar and v is the volume of the groove of the Lüders band. Using this equation to eliminate G from equation (9) and also making the substitution $\cos\theta = \sqrt{2}/\sqrt{3}$, equation (9) becomes

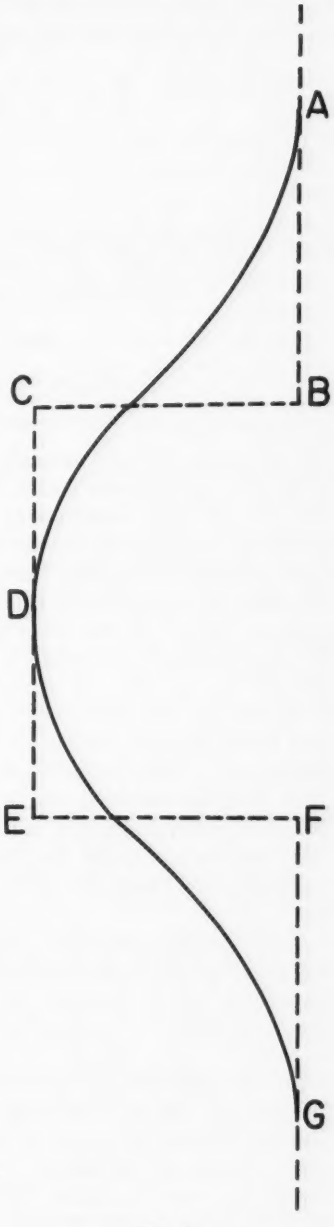


FIG. 2

$$\frac{v}{V} = \frac{(1 + b - \beta/2)b}{2(b - \beta/2)} \left(\frac{\tau - \bar{\tau}}{E}\right),$$

where V is the volume of the bar itself. Corresponding to the approximations (10) and (11), we now have the relations

$$\frac{v}{V} = \frac{1 + b}{2} \left(\frac{\tau - \bar{\tau}}{E}\right); \quad \frac{v}{V} = \frac{1}{2} \left(\frac{\tau - \bar{\tau}}{E}\right).$$

---

*THE RELATION BETWEEN X-RAY DOSAGE AND THE FREQUENCY OF SIMULATED HEALING OF CHROMOSOME BREAKAGES IN DROSOPHILA MELANOGASTER FEMALES**

BY IRWIN H. HERSKOWITZ

DEPARTMENT OF ZOOLOGY, INDIANA UNIVERSITY†

Communicated by H. J. Muller, May 12, 1954

X-radiation produces breakages in the proximal chromatin of attached-X chromosomes in germ cells of female *Drosophila*,[1] thus resulting in "detachment" of the X's. In the most commonly found type of case, detachment involves recombination of the heterochromatin of the X with the Y,[2] but Rapoport[3, 4] reported obtaining detached X's from females which had attached X's but contained no Y. Although he did not explain how he had achieved the production of attached-X females without a Y, they were later obtained by the cross of males of Lindsley and Novitski's stock, in which the X has both parts of the Y attached[5] to females with attached X's, and the production of detached X's by the resulting Y-less attached-X females was fully confirmed.[6, 7] The detached X's thus produced, when recovered in the next generation, are stable, containing a centromere, and can persist indefinitely. Presumably the acentric arms are lost. It is likely that the centric arms which are recovered represent only a fraction of those produced, since it is to be expected that sometimes a centric arm, after dividing, would have its daughter chromosomes join at the now-duplicated point of breakage, producing a dicentric isochromosome which eventually resulted in a bridge and thus in the death of the cell.

Two hypotheses have been presented to account for those cases derived from Y-less females in which the centric arm persisted. Either the broken ends healed, i.e., formed new stable telomeres,[3] or the cases simulating new telomere formation were actually the result of segmental interchange.[7] In the interchange process a telomere-bearing subterminal region of another chromosome (an autosome) would usually become joined to the centromere-bearing portion of the broken attached-X chromosome. In the case of the fourth chromosome, however, the portion which became attached to the arm of the X could be long enough to include, on occasion, the centromere of the fourth, but in that case the arm of the X involved would be the acentric one. Likewise, in a minority of cases involving chromosomes 2 or 3, the acentric arm of the X might become substituted for so small a subterminal re-

gion of the autosome that the deficiency of this region, simultaneously produced in heterozygous condition, would not kill the offspring having the "detached" X. The loss of the parts complementary to those surviving would in some cases occur by the breakage-fusion-bridge cycle and in others by their becoming segregated to a different pole from the surviving parts at some meiotic or mitotic division. In other papers[6–8] other evidence to test the adequacy of the interchange hypothesis is given, based on the nature of the products formed.

The present experiments were designed to attack the problem from a different angle. On the former hypothesis, that of adaptive telomere formation, the frequency of healed breakages would be expected to increase in a linear manner with increase in X-ray dosage, since the frequency of breaks themselves is proportional to dose, while according to the latter view, that of interchange, the frequency of rearrangements, each requiring the simultaneous occurrence of two or more breakages for their production, would be expected to increase as an exponent of the dose distinctly higher than 1 as is known to be the case with ordinary translocations.[9, 10] Accordingly, females were treated with two different dosages of X-rays, and the numbers of exceptional F_1 individuals of appropriate types were determined.

Materials and Methods.—Virgin females of the "snoc" attached-X stock, having no Y chromosome, carrying sc ct^{c} oc car on one of the arms of the X and y In49 sn^{s2} on the other, were used. The normal alleles of sc and y serve as markers for the ends of the X chromosome arms which carry them, except for a tiny terminal region distal to these markers (see Fig. 1). The females were collected within a period of 12 hours after their eclosion. After being aged for 3 days more, the females were divided into nine groups, one of which was used in control crosses, four of which were simultaneously given about 4000 r of X-rays, and the remaining four of which, all equal in number, were given about 1000 r by treating each one during a different quarter of the treatment time for the larger dose. By this means it was assured that the average X-ray dose administered was precisely four times less in the flies treated with the smaller dose than it was in those given the larger one. Immediately after irradiation the females were mated in bottles with y sc^{s1} B In49 v/Y^{+} males. The presence of recessive y and sc alleles here allow the dominant y^{+} and sc^{+} markers of the chromosome ends of the females' attached X's to manifest themselves in offspring bearing the father's X and part of the mother's attached X's. In an attempt to obtain not very different degrees of crowding, and therefore of selection for or against exceptions, in the cultures representing different doses, the numbers of female and male parents introduced per bottle were, respectively, 150 and 300 for 4000-r cultures, 30 and 60 for 1000-r cultures, and 17–30 and 30 for the controls. The parents were transferred to new bottles after two days and were discarded at the end of the fourth day after irradiation. The crosses were carried out at 25 ± 1° C.

Results.—The results of nine series of experiments, each containing treatments with 1000 r and 4000 r as well as controls, are summarized in Table 1. Figure 1 shows in simplified form the X-chromosome composition of the parental and F_1 flies, so far as shown by their phenotypes, but omits all representation of the autosomes and parts of autosomes which may have been involved. In the F_1 of the control crosses there were 5078 unexceptional "snoc" females and 47 females produced by nondisjunction in the male parent, but only 135 males (all unexceptional, having

UNIVERSITY OF MICHIGAN LIBRARIES

the paternal X). Even though one-half of the fertilized eggs regularly fail to develop, being composed of equal numbers of eggs with three X chromosomes and

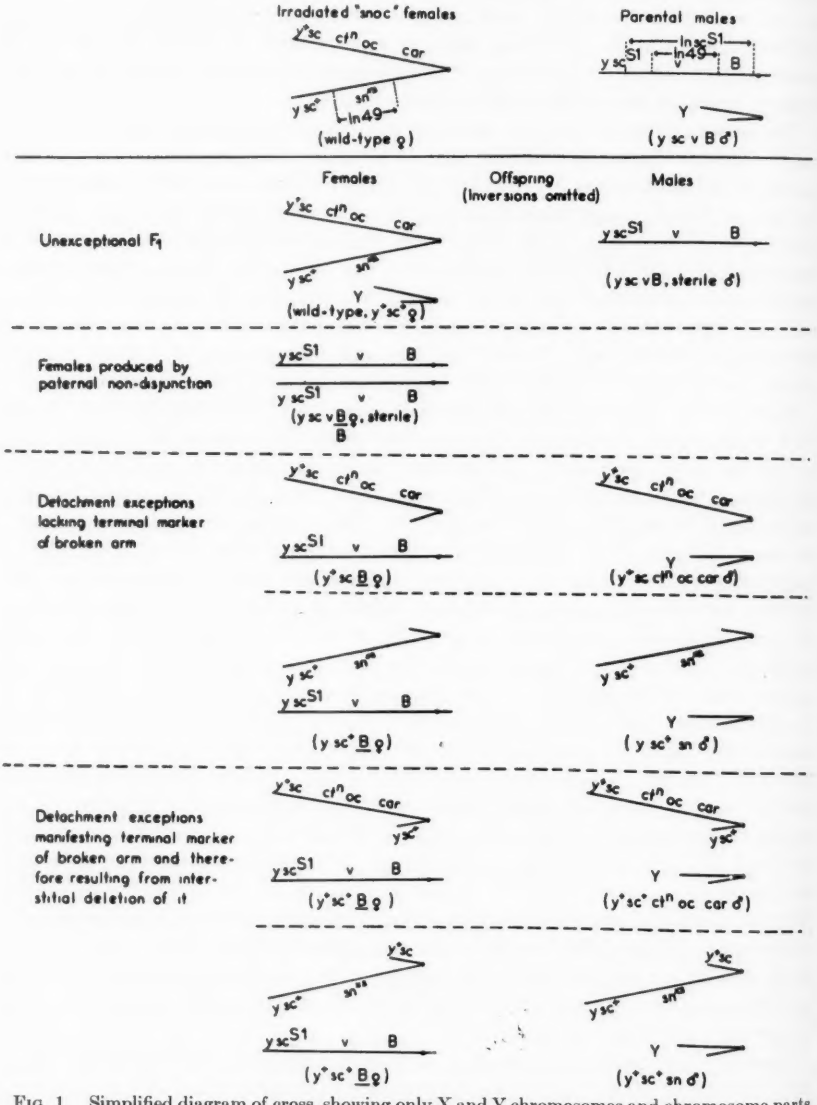


FIG. 1. Simplified diagram of cross, showing only X and Y chromosomes and chromosome parts Besides the mutant genes used as markers, their dominant normal alleles are also indicated, in the case of the loci y and sc. (Phenotypes are given in parentheses below gentoypes.)

eggs with none, the individuals in the one-half which may develop have very unequal chances for survival as adults. For, while the developing unexceptional F_1

females have a wild-type phenotype, the y sc^{S1} B v phenotype of the developing F_1 males is so detrimental that very few survive as adults after competition with their sisters. The mean number of 85 (5078/60) for unexceptional female survivors per bottle in the controls must represent an initial developing population of more than 170 individuals per bottle. No exceptional males or females, indicating spontaneous loss of part or all of an arm of the "snoc" attached-X, were obtained in the controls; nor were any triplo-X (wide-Bar) females found.

The developmental conditions in the bottles containing females treated with 1000 r were similar to those in the control bottles, since 90 unexceptional "snoc" F_1 fe-

TABLE 1

F_1 OFFSPRING FROM CROSSES OF y sc^{S1} B In49 v/Y^+ ♂♂ WITH "SNOC" ♀♀ WHICH WERE EITHER UNTREATED OR TREATED WITH 1000 r OR 4000 r OF X-RAYS

	Control			1000 r			4000 r		
	Brood 1	Brood 2	Total	Brood 1	Brood 2	Total	Brood 1	Brood 2	Total
No. bottles	30	30	60	35	35	70	33	33	66
Males:									
ct^8 oc*	0	0	0	0	0	0	1	0	1
sc ct^8 oc car†	0	0	0	3	6	9	0	6	6
y sn^{x2}†	0	0	0	2	0	2	2	1	3
Unexceptional (y sc^{S1} *B* In49 v)	62	73	135	102	157	259	29	52	81
Females:									
y sc^{S1} B In49 v (paternal X's)	15	32	47	9	34	43	0	15	15
Heterozygous Bar	0	0	0	15	27	42	23	39	62
Unexceptional ("snoc")	1895	3183	5078	1679	4635	6314	391	1000	1391
Per cent exceptional ♀♀ of all ♀♀ (excluding nondisjunctionals)	0	0	0	0.9	0.6	0.67	5.6	3.8	4.26

* Has the y^+ and sc^+ markers representing both ends of the attached X's.

† The class designated as sc ct^8 oc car ♂ included one individual not having car, while that designated as y sn^{x2} included 2 having car. This is because during the breeding of the "snoc" stock car sometimes changes places between the arms of the attached X's by crossing over between the right end of In49 and the locus of car, and these crossovers, being unrecognizable phenotypically, gradually accumulated in the stock.

males were produced per bottle. Eleven male exceptions representing detachment of the arms of the X's were found in the 1000-r bottles, in addition to the 259 unexceptional males obtained. Among the females, in addition to the 6314 unexceptional "snoc" F_1 females, there were 42 exceptions representing detachment and 43 produced by paternal nondisjunction. On the other hand, despite the attempt to get a number of offspring per 4000-r bottle that was similar to that obtained for the control and 1000-r bottles, by means of placing larger numbers of parents in the bottles, the average number of "snoc" F_1 females obtained was only 21 per 4000-r bottle. At this dosage, 10 exceptional males representing detachment were formed along with 81 unexceptional F_1 males, and 62 exceptional "detachment" females and 15 produced by paternal nondisjunction were obtained, besides 1391 "snoc" F_1 females.

The frequencies of exceptional individuals in relation to total females other than those produced by paternal nondisjunction, from the 1000-r and 4000-r treatments,

UNIVERSITY OF MICHIGAN LIBRARIES

are, among the males, 0.173 and 0.69 per cent, respectively, a ratio of 1:4, and among the females, 0.67 and 4.26 per cent, respectively, a ratio of 1:6.46. The phenotypes of the exceptional females and their frequencies are given in Table 2.

TABLE 2

TYPES AND FREQUENCIES OF HETEROZYGOUS BAR EXCEPTIONAL FEMALES

PHENOTYPE	1000 r		4000 r		4000 r PER CENT/ 1000 r PER CENT
	No.	Per Cent	No.	Per Cent	
y B	14	0.25	27	1.86	...
sc B	12	0.20	25	1.72	...
(Total)	(26)	(0.41 ± 0.081)	(52)	(3.58 ± 0.485)	8.8 ± 2.1
y^+ sc^+ B*	16	0.25 ± 0.063	10	0.69 ± 0.218	2.8 ± 1.1
TOTAL	42	0.67 ± 0.10	62	4.26 ± 0.53	6.46

* The presence of the markers representing both ends of the attached X's shows that in these cases there was an interstitial deletion of one arm.

Discussion.—Before considering the bearing which the frequencies of exceptional individuals following different X-ray treatments have on the mechanism of detachment, it is necessary to discuss how the variation in culture conditions may have influenced the results. Now the half-pint bottles used, containing about 50 cc. of a yeast-enriched food medium, provide the best culture conditions when some 200–300 flies are developing in them. When very many flies, say more than 500, develop in such a bottle, the amount of food is a limiting factor, causing delay in development, reduction in the size of the adults, and death to some of the developing individuals. When very few, say less than 50, animals develop in such a bottle, yeast, molds, and bacteria overgrow the culture so fast, because relatively few individuals are eating and churning the food, that the food pad becomes a very much poorer place to obtain nutrition and to complete development. In the present experiments, the numbers of offspring per bottle were much smaller in the 4000-r cultures than in the control or in the 1000-r cultures. Since the irradiation produces dominant and partially dominant lethal and detrimental mutations (especially, when we consider the attached-X females, aneuploid chromosome abnormalities), this would cause the frequency of the unexceptional males and paternal nondisjunctional females, relatively to the unexceptional females, to rise as the dose rose. The rise in their frequency in passing from the controls to the 1000-r group is, however, mainly an indication of better culture conditions in the latter (more optimal crowding), because that rise is much greater than the rise in their frequency in passing from the 1000-r group to the 4000-r group, even though the increase in dosage is far greater (3×) in the latter case. Now, since the 1000-r bottles produced (somewhat, though only slightly) more flies per bottle than the controls (but more if the dying larvae caused by irradiation are considered) yet had better culture conditions for development than the controls, because of the greater crowding, the 4000-r bottles must really have had *much* worse conditions than the 1000-r bottles, since they had *much* less crowding, as shown by the very small number of flies hatched per bottle.

Because the control and, more especially, the 1000-r cultures were grown under fairly good conditions, while the 4000-r cultures provided poorer conditions for development, one would expect this difference to be reflected in the numbers and types of flies which survived in the two cases. In favor of a better relative viability of "detachment" females over "snoc" females is the fact that the former each carry two X chromosomes, one of which has not been irradiated, while the latter carry

irradiated attached-X's. Therefore, certain types of induced X-chromosome mutations with dominant detrimental effects (mainly those types of aneuploidy in respect to which the two arms are not complementary) will occur oftener in the exceptional than in the unexceptional females. However, there are reasons for inferring that, despite this, "snoc" females have, on the average, viabilities that are superior to those of detachment females. For, while the former have a "wild-type" phenotype, the latter are all heterozygous Bar, some being also either yellow or scute. More important, regardless of which hypothesis for the origin of detachments obtains, all exceptional individuals are to some extent aneuploid and many are highly aneuploid, while this is much less often the case for the unexceptional ones. Moreover, with increase of dose this selective difference adverse to the exceptions would become greater.

For these reasons the frequency obtained by dividing the observed number of exceptional females by the sum of the observed numbers of "snoc" and exceptional females is very probably below the frequency which occurred during the egg stage, and the frequency would fall shorter, relatively to its true value, the higher the dose. Moreover, when the culture conditions are poorer (quite apart from the effect of increased dose in selectively damaging the flies), the value observed would be a greater underestimate of the true egg-stage value than when food conditions are better. Accordingly, it is probable that for both these reasons the observed percentage of exceptions has been reduced by adverse selection more at 4000 r than it has been at 1000 r. This would make the real difference between the frequencies of exceptional females at the two doses actually greater than observed. It may be added that certain tests of the data which have been made lend support to this interpretation.

One comparison of the incidence of surviving exceptional flies at 1000 r and 4000 r can be made, using the frequencies obtained, by dividing the numbers of exceptional males by the numbers of unexceptional and detachment females. It must first be pointed out that these males, because of their recessive markers, have much poorer viability than the unexceptional females and even than the detachment females. Second, the chance for survival of exceptional males is undoubtedly much better in 1000-r than in 4000-r cultures because of the afore-mentioned much better conditions for development in the former than in the latter group. Third, the radiation would have had a far more detrimental effect on the exceptional males, because of their haplo-X condition in which the X had been irradiated (that is, any genetic changes would, so to speak, lie uncovered), than on any of the other classes of flies, and this influence would reduce their collective viability increasingly at higher doses. Thus, to a far greater extent than in the case of the exceptional females, the difference in culture conditions and that in radiation both work in the same direction to reduce the frequency of exceptional males found, relative to that produced, much more at the higher than at the lower dose. The reckoning from the data gives, at 4000 r and at 1000 r, percentages of 0.69 and 0.173, respectively, a ratio of 4:1. In view of the foregoing discussion, the actual ratio would be much larger than this and therefore much larger than the 4:1 ratio of the doses of radiation.

A more nearly correct estimate of the frequency of exceptional flies in these experiments is obtained when the number of exceptional females is compared with

UNIVERSITY OF MICHIGAN LIBRARIES

the number of their unexceptional sisters. At 1000 r, 0.67 ± 0.10 per cent of the females were exceptions, while at 4000 r there were 4.26 ± 0.53 per cent. This ratio of 6.46:1 is higher than is consistent ($P \sim 0.02$, i.e., on a fiduciary level of about 1 in 50) with the 4:1 ratio expected, were all, or almost all, the exceptions consequent to single breakage events. This does not yet mean that all or almost all the exceptions were produced after multiple breakages. For, although we know from previous work[7] that one component of the group of exceptional females results from two-break cases, in which a subterminal part of the X, bearing the telomere, joins at its broken end to another breakage point located in or near the proximal heterochromatin of this chromosome, it would still seem possible that a detectable number of the exceptions might be the result of single breakages in or near the proximal heterochromatin followed by healing. If this were the case, then, upon separating the total group of exceptional females into two groups, one known to contain only intra-X interchanges and the other potentially composed, in part, of cases of single breakage followed by healing, additional analysis could be made.

The data in Table 2 show that the group of y B and sc B exceptions which potentially contains single healed breakages constitutes 3.6 ± 0.5 per cent at 4000 r and 0.41 ± 0.08 per cent of the females at 1000 r, being in the proportion of 8.8:1 for the two treatments. Such a ratio is statistically higher ($P \sim 0.01$) than the 4:1 ratio expected if all of these exceptions were the consequence of single breakages. Surprisingly enough, then, this group of exceptions, from which support for healed breakages would be most apparent, actually gives stronger evidence to the contrary than does consideration of the whole group of exceptions or of the group of exceptions known to be composed of intra-X interchanges. For this latter group increased from 0.25 ± 0.06 per cent at 1000 r to only 0.69 ± 0.22 per cent at 4000 r, a 2.8-fold increase, which is statistically consistent with ($P = 0.70$), although not evidence for, an increase which is linearly proportional to the X-ray dosage. That exceptions of fundamentally this type, i.e., large deletions of the X, increase in frequency as the $^3/_2$ power of the dose when sperm are irradiated was shown by Muller, Vogt, and Koerner.[10] Evidently, then, either the accident of small numbers or some special circumstance (see below) has caused the observed ratio in our experiment to fall below that which actually obtains. At any rate, no justification remains for considering any of the exceptional females as resulting from single healed breakages.

A possible explanation for the apparently nonexponential increase in the frequency of intra-X interchange females with dosage can be based on the supposition that the hyperploidy of these females, with an extra bit of the X subterminal region, tends to be more damaging than that of the y B and sc B exceptions, which have autosomal hyperploidy, thus resulting in a much lower viability, or at least on the supposition that there is a longer region of the X in question that has a sliding scale of viability (varying greatly with conditions). Consistent with this view is the observation[8] that a large proportion of the autosomal hyperploids involve chromosome 4, and, since one arm of this is entirely heterochromatic and the other known not to cause much inviability when present in triple dose, many of the autosomal hyperploids would be relatively immune to being selectively crowded out by poor cultural conditions. It has been noted in the foregoing that the poorer cultural conditions in the cultures for the higher dose than in those for the lower dose act selectively against flies carrying the most genetic detriment.

Two conclusions may be reached from these experiments. The first is that the data are consistent with the view that all the exceptions produced could well be the result of two or more breakages. This means that the earlier view[3] that a detectable percentage of new telomere formation is found after irradiating females, is no longer tenable, and therefore supports the alternative hypothesis[7] that the results could be due entirely to eucentric chromosomal interchanges.

It has been reckoned[7] that if the chromosomes, at the time the interchanges occurred, were in the tetrad stage, the frequency of interchanges which were phenotypically unidentified (those F_1 females in the present study having sc B or y B phenotypes) would be $9^1/_2$ times as great as that of interchanges which were identified as involving only the X (those represented in our present study by exceptions having y^+ sc^+ Bar phenotypes). This calculation assumed that the subterminal regions of all chromosomes are equally breakable and liable to be included in interchanges the products of which can reach maturity, that the frequency of breaks to the left of scute is so small as to be negligible for these purposes, and that the distribution of the chromosomes among egg and polar bodies is not substantially different for eucentrically interchanged chromosomes of the different types in question, involving the tips of different chromosomes. The calculation also ignored the reciprocal type of case, in which a detached acentric arm of an X became attached in place of the subterminal region of an autosome, which thereby became haplodeficient in that region (a condition severely restricting the size of the piece absent in a viable exception). If such cases, whatever their relative number, were added, the identifiable y^+ sc^+ B exceptions would be outnumbered that much more than $9^1/_2$ times. It seems, however, that one (or more) of the original postulates requires modification, for, although the real ratio of unidentified interchanges is probably higher than the 3:1 in the present experiments (Table 2), it, like the numbers 45:6 obtained in earlier experiments,[7] is probably less than $9^1/_2$:1.

The present data offer no evidence, except in those interchanges which result in the y^+ sc^+ B individuals, concerning the origin of the subterminal regions which joined to the break proximal to the centromere of the X. A number of possible explanations of the departure of the ratio of the unidentified to the identified interchanges from the "expected" $9^1/_2$:1 suggest themselves here, however. It may be that subterminal breakage of the X is higher than it is in the autosomes. More likely is the possibility that the former breaks oftener succeed in accomplishing interchanges with breaks in the proximal heterochromatin of the X than the latter breaks do. One ground for this would be provided if in oöcytes chromosomes were in such relative isolation from one another as to considerably favor intra- over interchromosomal gross rearrangements. Another ground (suggested by Muller) would be the presence of Inversion 49 in heterozygous condition in the attached X's, inasmuch as the loop thus caused would result in the distal region of the X being brought about twice as close to the proximal heterochromatin as the tips of the other chromosomes. Finally, elimination of the X's carrying subterminal pieces of autosomes into the polar bodies might be more frequent than that of X's with large deletions. At the same time, it must be recognized that viable cases of detached X's resulting from the substitution of an acentric arm of the X in place of the tip of an autosome must arise even less frequently than the complementary cases just

UNIVERSITY OF MICHIGAN LIBRARIES

considered—as is to be expected, because of the limitations on their viability already mentioned.

The second conclusion reached is that evidence has been obtained, for the first time, for a considerable number of gross chromosomal rearrangements taking place after *Drosophila* females are X-rayed. That interchanges are a frequent type of mutation in irradiated females is evident both from the present study, where more than 4 per cent of interchanges were produced at 4000 r, and earlier ones.[5, 7] Schultz[11] has summarized the almost completely negative results of other early studies to detect chromosome rearrangements after X-raying females. Glass[12] reported only six gross chromosomal aberrations in 2189 tests from females treated with about 2000 r of X-rays. In extensive analyses of changes involving specific loci, occurring in females, no spontaneous gross rearrangements were found,[13] nor were any found after treatment of oöcytes and oögonia with 4600 r, although later oöcytes gave a frequency of presumptive deficiencies (small deletions) at these loci similar to the results from treated spermatozoa and significantly higher than the results with oögonia.[14] The conclusion was reached[14] that chromosome breakage evidently occurs as frequently in late oöcytes as in spermatozoa but that union of fragments occurs fairly promptly, i.e., prior to the movements of the meiotic divisions, so that there is less opportunity for gross rearrangements to be formed.

In the present experiments only flies derived from late oöcytes have been studied. Having learned from the present work that the exceptions obtained earlier[7] were probably all interchanges between broken ends, it can now be stated definitely that the frequency of these interchanges, of gross rearrangement type, is higher when mature oöcytes are treated with X-rays than when younger oöcytes or oögonia are. This is consistent with the results obtained earlier for X-ray-induced deficiencies (small deletions) in these cells[14] but presents a contrast with the situation found in that and other work for gross rearrangements.

Probably the paucity of gross interchanges previously reported is due to the techniques of detection used, if such interchanges, when produced in females, are especially likely to involve breaks in subterminal and in heterochromatic regions or if it is especially likely that, in females, when two broken pieces join, the other two will fail to find and fuse with one another. Since it is exactly cases of these kinds which our own technique was adapted to detect, our data do not bear directly on the frequencies of the more orthodox types of gross interchanges.

The experiments in the present and in an earlier paper[7] do not determine whether or not the detached X's which showed the recessive marker (y or sc) of the distal end of one X represented deletions in which one break was to the left of the given locus, or interchanges with autosomes. Experiments designed to investigate this matter will be reported separately.[8]

Summary.—Irradiation of late oöcytes, having the "snoc" attached-X chromosome and containing no Y chromosome, with 1000 r and 4000 r of X-rays has produced a large number of cases in which the X was broken and one of the arms produced persisted in the offspring. The incidences of such individuals among the F_1 females at these doses were 0.67 ± 0.10 and 4.26 ± 0.53 per cent, respectively. Analysis of these data and of their components shows that there is no basis for believing that the broken chromosomes became stable by a process of new telomere formation following single breakage. The frequencies of those exceptions which

might according to their phenotype, have represented single breakages, being in the ratio of 1:8.8, clearly represent gross chromosomal rearrangements in which eucentric union occurred between one breakage point in or near the proximal heterochromatin on one of the arms of the X chromosome and another breakage point located subterminally, sometimes in the X, but probably more often in other, chromosomes. The present data, indicating that all the exceptional flies may be considered to carry gross rearrangements, make it possible to state definitely what was before only suggested[7]—that gross rearrangements occur more frequently after X-raying oöcytes than after X-raying oögonia. The fact that gross rearrangements have been detected only rarely in other studies after treating oöcytes is probably due to the unsuitability of the genetic methods employed to detect changes of the types occurring in this material.

* This work has been supported by a grant from the United States Atomic Energy Commission (Contract AT[11-1]-195). It is a pleasure to thank Mr. Seymour Abrahamson for his excellent assistance in many phases of this work.

† Contribution No. 557.

[1] H. J. Muller, and A. L. Dippel, *Brit. J. Exp. Biol.*, **3**, 85–122, 1926.

[2] U. Philip, *Nature*, **133**, 726, 1934.

[3] J. A. Rapoport, *Compt. rend. acad. sci. U.R.S.S.*, N.S., **29**, 612–615, 1940.

[4] *Ibid.*, **31**, 266–269, 1941.

[5] D. L. Lindsley, Jr., and E. Novitski, *Drosophila Information Service*, **24**, 84, 1950.

[6] I. H. Herskowitz, and H. J. Muller, *Records Genet. Soc. Amer.*, **22**, 79, 1953; and *Genetics*, **38**, 669, 1953.

[7] H. J. Muller, and I. H. Herskowitz, *Amer. Naturalist*, 1954 (in press).

[8] H. J. Muller, S. Abrahamson, and I. H. Herskowitz (in manuscript).

[9] M. L. Belgovsky, *Trudy Inst. Genet.* (Moscow), **11**, 93–124, 1937; H. J. Muller, *J. Genet.*, **40**, 1–66, 1940.

[10] H. J. Muller, *Collecting Net* (Woods Hole), **13**, 181, 183–195, 198, 1938.

[11] J. Schultz, *Biological Effects of Radiation* (London: McGraw-Hill Book Co., 1936, Vol. **2**, 1209–1261 (chap. xxxix).

[12] B. Glass, *Records Genet. Soc. Amer.*, **8**, 117, 1939; and *Genetics*, **25**, 118, 1940; B. Glass, *Records Genet. Soc. Amer.*, **18**, 89–90, 1949; and *Genetics*, **35**, 109–110, 1950; B. Glass, *Records Genet. Soc. Amer.*, **21**, 28–29; and *Genetics*, **37**, 584–585, 1952.

[13] H. J. Muller, J. I. Valencia, and R. M. Valencia, *Records Genet. Soc. Amer.*, **18**, 105–106, 1949; and *Genetics*, **35**, 125–126, 1950.

[14] H. J. Muller, R. M. Valencia, and J. I. Valencia, *Records Genet. Soc. Amer.*, **18**, 106, 1949; and *Genetics*, **35**, 126, 1950.

UNIVERSITY OF MICHIGAN LIBRARIES

ON TORSION IN LIE GROUPS

By Raoul Bott

UNIVERSITY OF MICHIGAN

Communicated by Marston Morse, May 4, 1954

1. G shall denote a semisimple compact connected and simply connected Lie group. In Theorem I we show how the "diagram" of G completely determines the integral homology groups of $\Omega(G)$ (the space of loops on G). In Theorem II we prove the same proposition concerning $H(G/T)$, where T is a maximal torus of G. Immediate corollaries are:

(1) *$\Omega(G)$ and G/T have no torsion. Their Betti numbers vanish in odd dimensions.*

(2) *If G is simple, then $\pi_3(G) \approx Z$.*

Most of these results are known for the classical groups, as well as G_2 and F_4. They are contained in Borel's work.[1] They are new, I believe, for the exceptional groups E_6, E_7, and E_8. The proof of both theorems makes no use of the classification of Lie groups. Both are proved via the Morse theory.[2] Both are straightforward consequences of the "lacunary" principle of Morse, which, for our purpose, we state as follows: If J is a smooth function on a compact Riemann manifold M, and if the type numbers M_n of J on M are zero for odd n, then M is free of torsion and $M_n = R_n(M)$—the nth Betti number of M.

This proposition follows immediately from the Morse inequalities, which are seen to imply that $M_n = R_n(M)$ for any coefficient field. This principle is equally applicable to the type numbers of the length function J on the loop space $\Omega(M)$.

We will sketch the proofs of both theorems, assuming familiarity with the Morse theory[2] and the geometrical knowledge of G, due in large part to É. Cartan.[3]

2. We assume from now on that G is simple. An element of T is called singular if the dimension of its normalizer is greater than the dimension of T. Let E be the universal covering group of T and let $p: E \to T$ be the covering projection. The set of points S in E which map onto singular points of T form several families of parallel planes in E which divide E into simplexes $\{\Delta\}$. (This partitioning of the Euclidean space E is the "diagram" of G.)

The points in the kernel of p make up a lattice L in E, and every simplex Δ has precisely one vertex in L because G is simply connected.

Let $\tilde{e}$ be a fixed point of L, $\bar{\Delta}$ a fixed simplex containing $\tilde{e}$. Let C—a fundamental chamber of G—be the convex cone of points $x = \Sigma\alpha_i x_i$, $\alpha_i \geq 0$, $x_i \in \bar{\Delta}$.

If x is a point of E, we define its index $\lambda(x)$ to be equal to twice the number of singular planes crossed by the straight line joining $\tilde{e}$ to x. λ is constant in the interior of each simplex Δ in E. We denote this constant value by $\lambda(\Delta)$. Let $Q(C; t) = \sum_{\Delta \subset C} t^{\lambda(\Delta)}$; $Q(x; t) = \sum_{\Delta \subset C;\ x \in \Delta} t^{\lambda(\Delta)}$, for $x \in C$. Finally, if h is a direction at e on G, let G_h be the subgroup of G which leaves h invariant under the adjoint operation.

Theorem I. *$\Omega(G)$ is free of torsion. The Poincaré polynomial of $\Omega(G)$ is given by*

$$P(\Omega(G); t) = Q(C; t),$$

where C is a fundamental chamber of G.

Theorem I is proved as follows: Let $\bar{x} \in \bar{\Delta}$ be a fixed interior point of $\bar{\Delta}$. Let $\{\bar{x}_\alpha\} \in E$ be the set of points of E which cover $x = p(\bar{x})$ in T. Let $\{\bar{s}_\alpha\}$ be the straight lines in E joining $\bar{e}$ to $\bar{x}_\alpha$. Let $\{s_\alpha\}$ be their projections: $p(\bar{s}_\alpha)$. Then (1) the set $\{s_\alpha\}$ constitutes a complete system of geodesic arcs joining e to x in G; (2) the conjugate points of e on s_α correspond to the intersection of $\bar{s}_\alpha$ with a singular plane, each plane giving rise to a conjugate point of weight two. Hence the geodesics $\{s_\alpha\}$ are all nondegenerate, and their index is given by $\lambda(s_\alpha) = \lambda(\bar{x}_\alpha)$. Let M_n = number of geodesics joining e to x in G of index n. As M_n is 0 if n is odd, we obtain the first part of Theorem I, by applying the lacunary principle to $\Omega(e, x)$. But by an obvious symmetry argument we can shift (by operations of the Weyl group) the points $\{\bar{x}_\alpha\}$ into our fundamental chamber C, without affecting their indices. Every simplex of C will then contain precisely one of these points. Hence Theorem I.

THEOREM II. *G/T is torsion-free. The Poincaré polynomial of G/T is given by*

$$P(G/T; t) = Q(q; t)/t^{\lambda(q)},$$

where q is any lattice point in the interior of C.

Proof. Let $\mathfrak{L}$ denote the Lie algebra of G, equipped with the Euclidean metric derived from the adjoint representation.[3] E—the tangent space of T—is contained in $\mathfrak{L}$ and forms a Cartan subalgebra of $\mathfrak{L}$. Let x_0 be a fixed vector in the interior of the fundamental chamber C of E. Consider the map $F: G \to \mathfrak{L}$ defined by $F(g) =$ (Adg.) x_0.[4] The image of F is G/T imbedded in $\mathfrak{L}$. Let $y_0 \neq x_0$ be some other interior point of C. Let f be the function on G/T which assigns to the coset gT the distance of $F(g)$ from y_0. Let $\{x_\beta\}$ be the set of points obtained from x_0 by operations of the Weyl group, that is, the points obtained from x_0 by reflections in the planes $\{C^*\}$ of the diagram which pass through $\bar{e}$. We now claim:

LEMMA. *The set of points $\{x_\beta\} \subset F(G)$ are the only critical points of f on G/T. They are all nondegenerate, and the index of x_β is equal to twice the number of planes of the set $\{C^*\}$ intersected by a straight line joining x_β to y_0.*

This lemma proves Theorem II modulo some symmetry arguments.

The proof of the lemma runs as follows: If $c \in G/T$ is a critical point of f, then the line from y_0 to c will meet the manifold G/T at right angles. This implies $(y_0 - c, [h, c_0]) = 0$ for all h in $\mathfrak{L}$. As y_0 is a "general element" of $\mathfrak{L}$, $y_0 - c$ is contained in E. Hence c is one of the points of $\{x_\beta\}$. To find the index of f on x_β we study the focal points of G/T on the line from x_β to y_0. By the Jacoby principle, these are easily seen to occur at the intersection of that line with the singular planes $\{C^*\}$, each intersection having weight two.

When we apply the Morse theory to $\Omega(e, e)$, i.e., the space of closed loops on G, the critical sets will occur in manifolds corresponding to lattice points $\{q\}$ of C. When q is in the interior of C, these manifolds of critical points will be homeomorphic to G/T. By the use of the notion of a nondegenerate critical manifold,[5] a limiting argument can be used to prove the following refinement of Theorem II.

THEOREM II′. *Let q be a lattice point of C, and let h be its direction from $\bar{e}$. Then G/G_h is free of torsion, and*

$$P(G/G_h; t) = Q(q; t)/t^{\lambda(q)}.$$

Other applications and details will be reported elsewhere.

UNIVERSITY OF MICHIGAN LIBRARIES

[1] A. Borel, "Sur la cohomologie des espaces fibres principaux et des espaces homogènes des groupes de Lie compacts," *Ann. Math.*, **57**, No. 1, 115, 1953; "Homology and Cohomology of Compact Connected Lie Groups," these PROCEEDINGS, **39**, No. 11, 1142–1146, 1953.

[2] M. Morse, "The Calculus of Variations in the Large," *Math. Colloq. Pub.*, **18**, 1934.

[3] É. Cartan, "La Géometrie des groupes simples," *Œuvres*, **2**, Part I, 793; also papers referred to therein. See also E. Stiefel, "Über eine Beziehung. . ." *Comm. Math. Helv.*, **14**, 350–380, 1942.

[4] C. Chevalley, *Theory of Lie Groups I* (Princeton, N.J.: Princeton University Press, 1946).

[5] R. Bott, "Non-degenerate Critical Manifolds" (to appear in *Ann. Math.*).

---

*VALUATION EQUILIBRIUM AND PARETO OPTIMUM**

BY GERARD DEBREU

COWLES COMMISSION FOR RESEARCH IN ECONOMICS

Communicated by John von Neumann, May 6, 1954

For an economic system with given technological and resource limitations, individual needs and tastes, a valuation equilibrium with respect to a set of prices is a state where no consumer can make himself better off without spending more, and no producer can make a larger profit; a Pareto optimum is a state where no consumer can be made better off without making another consumer worse off. Theorem 1 gives conditions under which a valuation equilibrium is a Pareto optimum. Theorem 2, in conjunction with the Remark, gives conditions under which a Pareto optimum is a valuation equilibrium. The contents of both theorems (in particular that of the first one) are old beliefs in economics. Arrow[1] and Debreu[2] have recently treated this question with techniques permitting proofs. A synthesis of their papers is made here. Their assumptions are weakened in several respects; in particular, their results are extended from finite dimensional to general linear spaces. This extension yields as a possible immediate application a solution of the problem of infinite time horizon (see sec. 6). Its main interest, however, may be that by forcing one to a greater generality it brings out with greater clarity and simplicity the basic concepts of the analysis and its logical structure. Not a single simplification of the proofs would indeed be brought about by restriction to the finite dimensional case.

As far as possible the mathematical structure of the theory has been dissociated from the economic interpretation, to be found in brackets.

1. *The Economic System.*—Let L be a linear space (on the reals R).[3] The economic system can be described as follows:

The ith consumer ($i = 1, \ldots, m$) chooses a point x_i [his consumption] in a given subset X_i [his consumption-set] of L. [x_i completely describes the quantities of commodities he actually consumes, to be thought of as positive, and the quantities of the various types of labor he produces, to be thought of as negative. X_i is determined by constraints of the following types: quantities of commodities consumed (labor produced) must be nonnegative (nonpositive), and, moreover, they must enable the individual to survive.] There is on X_i a complete ordering, denoted by $\underset{i}{\leqq}$ [corresponding to the preferences of that consumer].[4] x_i^0 is a saturation point of X_i, if, for all $x_i \in X_i$, one has $x_i \underset{i}{\leqq} x_i^0$.

The jth producer ($j = 1, \ldots, n$) chooses a point y_j [his production] in a given subset Y_j [his production-set] of L. [y_j is a complete description of all his outputs, to be thought of as positive, and his inputs, to be thought of as negative. Y_j is determined by technological limitations.]

Denote $x = \sum_i x_i$, $y = \sum_j y_j$; they are constrained to satisfy the equality $x - y = \zeta$, where ζ is a given point of L. [ζ corresponds to the exogenous resources available (including all capital existing at the initial date). $x - y$ is the *net* consumption of all consumers and all producers together. It must clearly equal ζ.][5]

A $(m + n)$-tuple $[(x_i), (y_j)]$, one x_i for each i, one y_j for each j, is called a state of the economy. [It is a complete description of the activity of every consumer and every producer.] A state $[(x_i), (y_j)]$ is called attainable if $x_i \in X_i$ for all i, $y_j \in Y_j$ for all j, $x - y = \zeta$.

2. *Valuation Equilibrium.*—$v(z)$ will denote a (real-valued) linear form on L.[6] [It gives the value of the commodity-point z. When L is suitably specialized, this value can be represented by the inner product $p \cdot z$, where p is the price system.] A state $[(x_i^0), (y_j^0)]$ is a valuation equilibrium with respect to $v(z)$ if:

(2.1) $[(x_i^0), (y_j^0)]$ is attainable.

(2.2) For every i « $x_i \in X_i$, $v(x_i) \leqq v(x_i^0)$ » implies « $x_i \underset{i}{\leqq} x_i^0$ ». [Best satisfaction of preferences subject to a budget constraint.]

(2.3) For every j « $y_j \in Y_j$ » implies « $v(y_j) \leqq v(y_j^0)$ ». [Maximization of profit subject to technological constraints.]

3. *Pareto Optimum.*—The set $X_1 \times \ldots \times X_m$ of m-tuples (x_i), one x_i for each i, is (partially) ordered as follows: $(x_i') \geqq (x_i)$ if and only if $x_i' \underset{i}{\geqq} x_i$ for all i.

A state $[(x_i^0), (y_j^0)]$ is a Pareto optimum if:

(3.1) $[(x_i^0), (y_j^0)]$ is attainable.

(3.2) There is no attainable state $[(x_i), (y_j)]$ for which $(x_i) > (x_i^0)$. [It is impossible to make one consumer better off without making another one worse off.]

4. *A Valuation Equilibrium Is a Pareto Optimum.*—The following assumptions will be made:

I. *For every i, X_i is convex.*

II. *For every i,* « $x_i' \in X_i$, $x_i'' \in X_i$, $x_i' \underset{i}{<} x_i''$ » *implies* » $x_i' \underset{i}{<} (1 - t)\, x_i' + t x_i''$ *for all t, $0 < t < 1$* ».

These two axioms on the convexity of the consumption-sets and the convexity of preferences have been used by Arrow and Debreu[7] in a different context.

THEOREM 1. *Under assumptions I and II, every valuation equilibrium $[(x_i^0), (y_j^0)]$, where no x_i^0 is a saturation point, is a Pareto optimum.*

Proof: (4.1) « $x_i \in X_i$ *and* $x_i \underset{i}{>} x_i^0$ » *implies* « $v(x_i) > v(x_i^0)$.»

This is a trivial consequence of definition (2.2).

(4.2). « $x_i \in X_i$ *and* $x_i \underset{i}{\sim} x_i^0$ » *implies* « $v(x_i) \geqq v(x_i^0)$.»

Since x_i^0 is not a saturation point, there is $x_i' \in X_i$, such that $x_i' \underset{i}{>} x_i^0$, hence $x_i' \underset{i}{>} x_i$. Consider $x_i(t) = (1 - t)\, x_i + t x_i'$. By assumption II, for all t, $0 < t < 1$, $x_i(t) \underset{i}{>} x_i$, hence $x_i(t) \underset{i}{>} x_i^0$, so (by [4.1]) $v(x_i^0) < v(x_i(t)) = (1 - t)\, v(x_i) + t v(x_i')$. Let t tend to zero; in the limit $v(x_i^0) \leqq v(x_i)$.

To complete the proof we consider a state $[(x_i), (y_j)]$, where $x_i \in X_i$ for all i, $y_j \in Y_j$ for all j, and show that if $(x_i) > (x_i^0)$, the state is not attainable, i.e., $x - y \neq \zeta$.

$(x_i) > (x_i^0)$ means that for all i, $x_i \underset{i}{\geqq} x_i^0$, and for some i', $x_{i'}^0 \underset{i'}{>} x_{i'}^0$; so by (4.1) and (4.2) $\sum_i v(x_i) > \sum_i v(x_i^0)$, i.e., $v(x) > v(x^0)$. On the other hand, (2.3) implies $v(y) \leqq v(y^0)$, so $v(x) - v(y) > v(x^0) - v(y^0)$. Since $x^0 - y^0 = \zeta$, $v(x - y) > v(\zeta)$, which rules out $x - y = \zeta$.

5. *A Pareto Optimum Is a Valuation Equilibrium.*—In this section L is a topological linear space.[8] Let x_i', x_i'' be points of X_i; we define $I(x_i', x_i'') = \{t \mid [(1 - t)x_i' + tx_i''] \in X_i\}$. When X_i is convex, $I(x_i', x_i'')$ is a real interval with possibly one or two end-points excluded. In addition to assumptions I and II, three further assumptions are needed here.

III. *For every i, x_i, x_i', x_i'' in X_i the sets $\{t \in I(x_i', x_i'') \mid (1 - t)x_i' + tx_i'' \underset{i}{\geqq} x_i\}$ and $\{t \in I(x_i', x_i'') \mid (1 - t)x_i' + tx_i'' \underset{i}{\leqq} x_i\}$ are closed in $I(x_i', x_i'')$.*

This weak axiom of continuity for preferences has been introduced by Herstein and Milnor[9] in another context. We define $Y = \sum_j Y_j$ (the set of all $y = \sum_j y_j$, where $y_j \in Y_j$ for all j).

IV. *Y is convex.* [The assumption that the aggregate production-set is convex is strictly weaker than the assumption that the individual production-sets Y_j are all convex.]

V. *L is finite dimensional and/or Y has an interior point.* [The assumption that Y has an interior point will be shown in section 6 to be implied by free disposal of commodities.]

THEOREM 2. *Under assumptions I–V, with every Pareto optimum $[(x_i^0), (y_j^0)]$, where some x_i^0 is not a saturation point, is associated a (nontrivial) continuous linear form $v(z)$ on L such that:*

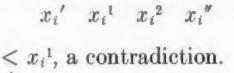
(5.1) *For every i " $x_i \in X_i$, $x_i \underset{i}{\geqq} x_i^0$ " implies " $v(x_i) \geqq v(x_i^0)$."*

(5.2) *For every j " $y_j \in Y_j$ " implies " $v(y_j) \leqq v(y_j^0)$."*

Proof: From assumptions I, II, and III follows:

a) " *x_i', x_i'' in X_i, $x_i' \underset{i}{\leqq} x_i''$* " *implies* " *for all t, $0 \leqq t \leqq 1$, $x_i' \underset{i}{\leqq} [(1 - t)x_i' + tx_i'']$ $\in X_i$.*"

By assumption III, the set $\{t \in I(x_i', x_i'') \mid (1 - t)x_i' + tx_i'' \underset{i}{<} x_i'\}$ is open in $I(x_i', x_i'')$. Its intersection with the interval $]0, 1[$ (end-points excluded) is open. We wish to show that this intersection is empty. If it were not, it would contain two numbers $t_1 < t_2$. Take the corresponding points x_i^1, x_i^2. Then $x_i^1 \underset{i}{<} x_i' \underset{i}{\leqq} x_i''$. By assumption II, $x_i^1 \underset{i}{<} x_i''$ gives $x_i^1 \underset{i}{<} x_i^2$.



Similarly, $x_i^2 \underset{i}{<} x_i'$ gives $x_1^2 \underset{i}{<} x_i^1$, a contradiction.

As an immediate consequence of *(a)*, for all i, the sets $X_{i(x_i^0)} = \{x_i \in X_i \mid x_i \underset{i}{\geqq} x_i^0\}$ and $\mathring{X}_{i(x_i^0)} = \{x_i \in X_i \mid x_i \underset{i}{>} x_i^0\}$ are convex.

Let i' be a value of i for which $x_{i'}^0$ is not a saturation point, and consider the set

$$Z = \overset{\circ}{X}_{i'(x_{i'}^0)} + \sum_{i \neq i'} X_{i(x_i^0)} - \sum_j Y_j.$$

$\zeta \notin Z$, this is the definition of a Pareto optimum $[(x_i^0), (y_j^0)]$. Z is convex as it is the sum of convex sets. If $Y = \sum_j Y_j$ has an interior point, Z also has one. The Hahn-Banach theorem[10] can therefore be applied to Z and ζ. There is a (non-trivial) continuous linear form $v(z)$ on L such that $v(z) \geqq v(\zeta)$ for all $z \in Z$, i.e., since

$$\zeta = \sum_i x_i^0 - \sum_j y_j^0, \qquad v\left[\sum_i (x_i - x_i^0) - \sum_j (y_j - y_j^0)\right] \geqq 0$$

for all $x_{i'} \in \overset{\circ}{X}_{i'(x_{i'}^0)}$, $x_i \in X_{i(x_i^0)}$ (for $i \neq i'$), $y_j \in Y_j$ (for all j).

In this statement $\overset{\circ}{X}_{i'(x_{i'}^0)}$ can be replaced by $X_{i'(x_{i'}^0)}$, for every $x_{i'} \in X_{i'}$, $x_{i'} \underset{i'}{\sim} x_{i'}^0$, can be exhibited, as in the proof of (4.2), as a limit of points belonging to $\overset{\circ}{X}_{i'(x_{i'}^0)}$. Therefore,

b) $\sum_i v(x_i - x_i^0) + \sum_j v(y_j^0 - y_j) \geqq 0$ for all $x_i \in X_i(x_i^0)$, $y_j \in Y_j$.

By making all but one of the x_i, y_j equal to the corresponding x_i^0, y_j^0, one proves that for the remaining term in *(b)* $v(x_i - x_i^0) \geqq 0$ for all $x_i \in X_i(x_i^0)$ (or $v(y_j^0 - y_j) \geqq 0$ for all $y_j \in Y_j$) which is precisely the statement of Theorem 2.

(5.2) is identical to (2.3), but (5.1) does not necessarily imply (2.2), and Theorem 2 does not quite correspond to the title of this section. The following Remark, due to Arrow[11] in its essence, tries to fill this gap:

REMARK. *Under assumptions I and III, if there is, for every i, an $x_i' \in X_i$ such that $v(x_i') < v(x_i^0)$, then* (5.1) *implies* (2.2).

Consider an $x_i \in X_i$, $v(x_i) \leqq v(x_i^0)$. Let $x_i(t) = (1 - t)\, x_i + t x_i'$. For all t, $0 < t < 1$, $v(x_i(t)) < v(x_i^0)$ and thus, by (5.1), $x_i(t) \underset{i}{<} x_i^0$. The set $\{t \in I\,(x_i, x_i') \mid (1 - t)\, x_i + t x_i' \underset{i}{\leqq} x_i^0\}$ contains the interval $]0, 1[$; since it is closed in $I(x_i, x_i')$ (by assumption III), it contains 0, i.e., $x_i \underset{i}{\leqq} x_i^0$.

[The condition that there is $x_i' \in X_i$ such that $v(x_i') < v(x_i^0)$ means that the consumer does not have such a low $v(x_i^0)$ that with any lower value he could not survive.]

6. *The Free Disposal Assumption.*—An example will show the economic justification of assumption V when L is not finite dimensional. Suppose that there is an infinite sequence of commodities [because, for example, economic activity takes place at an infinite sequence of dates, a case studied by Malinvaud[12] with different techniques]. The space L will be the set of infinite sequences of real numbers (z_h) such that $\mathrm{Sup}\,|z_h| < +\infty$. L is normed by $\|z\| = \mathrm{Sup}\,|z_h|$.

The assumption of free disposal for the technology means that if $y \in Y$ and $y_h' \leqq y_h$ for all h, then $y' \in Y$ [if an input-output combination is possible, so is one where some outputs are smaller or some inputs larger; it is implied that a surplus can be freely disposed of]. With this assumption, if Y is not empty, it clearly has an interior point: select a number $\rho > 0$ and a point $y \in Y$; consider y' defined by $y_h' = y_h - \rho$ for all h. The sphere of center y', radius ρ, is contained in Y.

Other examples of linear spaces in economics are provided by the case where there

is a finite number l of commodities, and time and/or location is a continuous variable. The activity of an economic agent is then described by the l rates of flow of the commodities as functions of time and/or location. The space L is the set of l-tuples of functions of the continuous variable.

In any case, if L is properly chosen, the existence of an interior point for Y will follow from the free disposal assumption. Then application of Theorem 2 will give a continuous linear form $v(z)$.

* Based on Cowles Commission Discussion Paper, Economics, No. 2067 (January, 1953). This article has been prepared under contract Nonr-358(01), NR 047-006 between the Office of Naval Research and the Cowles Commission for Research in Economics.

I am grateful to E. Malinvaud, staff members and guests of the Cowles Commission, in particular I. N. Herstein, L. Hurwicz, T. C. Koopmans, and R. Radner for their comments.

[1] K. J. Arrow, "An Extension of the Basic Theorems of Classical Welfare Economics," *Proceedings of the Second Berkeley Symposium* (Berkeley: University of California Press, 1951), pp. 507–532.

[2] G. Debreu, "The Coefficient of Resource Utilization," *Econometrica*, **19**, 273–292, 1951.

[3] A real linear space L is a set where the addition of two elements $(x + y)$ and the multiplication of a real number by an element (tx) are defined and satisfy the eight axioms:

1. For all x, y, z in L, $(x + y) + z = x + (y + z)$.
2. There is an element $0 \in L$ such that for every $x \in L$, $x + 0 = x$.
3. For every $x \in L$, there is an $x' \in L$ such that $x + x' = 0$.
4. For all x, y in L, $x + y = y + x$. For all x, y in L, t, t' in R,
5. $t(x + y) = tx + ty$,
6. $(t + t')x = tx + t'x$,
7. $t(t'x) = (tt')x$,
8. $1x = x$.

[4] An order is a reflexive and transitive binary relation (generally denoted by $\leqq$). $x \sim x'$ means $x \leqq x'$ and $x' \leqq x$, while $x < x'$ means $x \leqq x'$ and not $x' \leqq x$. The order is complete (as opposed to partial) if for any x, x' one has $x \leqq x'$ and/or $x' \leqq x$.

One may object to completeness of the preference ordering as well as to its transitivity. The reader must therefore note that, with slight modifications of the definitions and the assumptions, Theorems 1 and 2 can easily be proved for *arbitrary* binary relations on the X_i.

[5] Usually the net consumption is only constrained to be at most equal to the available resources. But this implies that any surplus can be freely disposed of. Such an assumption on the technology should be made explicit (see sec. 6) while requiring at the same time $x - y = \zeta$.

[6] For all x, y, $v(x + y) = v(x) + v(y)$. For all t, x, $v(tx) = tv(x)$. $v(z)$ is said to be trivial if it vanishes everywhere.

[7] K. J. Arrow, and G. Debreu, "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, 1954.

[8] A topological linear space is a linear space with a topology such that the functions $(x, y) \to x + y$ from $L \times L$ to L and $(t, x) \to tx$ from $R \times L$ to L are continuous. For definition of a topology, of the topology on a product, of a continuous function see N. Bourbaki, *Eléments de mathematique* (Paris: Hermann, et Cie, 1940), Part I, Book 3, chap. i. For the representation of continuous linear forms on L see S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932), in particular, chap. iv, sec. 4.

[9] I. N. Herstein and J. Milnor, "An Axiomatic Approach to Measurable Utility," *Econometrica*, **21**, 291–297, 1953.

[10] In a real topological linear space, if Z is a convex set with interior points, ζ a point which does not belong to Z, there is a closed hyperplane through ζ, bounding for Z. (See for example, N. Bourbaki, *Eléments de mathématique* [Paris: Hermann et Cie, 1953], Part I, Book 5, chap. ii, in particular, sec. 3.)

[11] Arrow, *op. cit.*, Lemma 5.

[12] E. Malinvaud, "Capital Accumulation and Efficient Allocation of Resources," *Econometrica* **21**, 233–268, 1953.

INTERDEPENDENCE OF THE YIELD CONDITION AND THE STRESS-STRAIN RELATIONS FOR PLASTIC FLOW

By T. Y. Thomas

NAVAL RESEARCH LABORATORY, WASHINGTON, D.C.

Communicated May 18, 1954

1. *The Fundamental Assumptions.*—Denote by $s_{\alpha\beta}$ the components of the deviation s of the stress tensor σ. Let v_α be the components of the velocity vector v and $\epsilon_{\alpha\beta}$ the components of the rate of strain tensor ϵ derived from the velocity vector. We impose the usual condition that the flow is incompressible, i.e., $\epsilon_{\alpha\alpha} = 0$, which appears to be in agreement with experimental evidence. Hence ϵ is identical with its deviation.[1] We now make the following fundamental assumptions:

I. *The stress deviation s is an invariant of the tensor ϵ under proper orthogonal co-ordinate transformations.* In other words we have

$$s_{\alpha\beta} = \Phi_{\alpha\beta}(\epsilon_{11}, \epsilon_{12}, \ldots, \epsilon_{33}), \tag{1}$$

where the quantities $\Phi_{\alpha\beta}$ are the components of a symmetric tensor invariant of the tensor ϵ relative to the proper orthogonal group. Relations (1) will be called the *stress-strain relations.* In addition to the components $\epsilon_{\alpha\beta}$ which appear explicitly in the right-hand members of (1), the quantities Φ may also involve, subject to their invariant character, point functions, e.g., scalar functions of the co-ordinates, and indeed terms depending, in some sense, on the strain history of the material, which is known to influence present behavior. The restriction to rectangular co-ordinates which is implied by Assumption I is natural and constitutes no loss of generality, since one can readily express relations (1), as well as all following relations, in arbitrary co-ordinates by the ordinary methods of the tensor analysis. Use of the deviation tensor s rather than the actual stress tensor σ in (1) has its justification in the experimental fact that plastic flow is influenced but little, if at all, by the effects of hydrostatic pressure; the precise argument which is made in this connection is well known and will not be repeated. Relations (1) include, in particular, the stress-strain relations of the von Mises plasticity theory. They do not strictly contain the Prandtl-Reuss stress-strain relations, since these involve, in addition to the above tensors s and ϵ, a tensor whose components are the time derivatives of the components of the stress deviation. By extending Assumption I to include this latter tensor, we can deduce the dependence of the yield condition on general stress-strain relations of the Prandtl-Reuss type.

II. *Relations* (1) *do not establish a* (1, 1) *correspondence between the components of the s and ϵ tensors.* There appears to be good experimental evidence for this assumption, which may, in fact, be regarded as expressing the essential distinction between plastic flow and the flow of ordinary fluids.

2. *Scalar Invariants.*—At the origin of a system of canonical co-ordinates for the tensor ϵ, the components $\epsilon_{\alpha\beta}$ of this tensor, relative to the canonical system, have the form $\epsilon_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$, in which there is evidently no summation on the index α. The λ_α are the so-called principal values or *principal invariants of* ϵ and are given as the three solutions λ, which are necessarily real, of the determinantal equation

$$|\epsilon_{\alpha\beta} - \lambda\delta_{\alpha\beta}| = -\lambda^3 + \frac{1}{2}\xi\lambda + \frac{1}{3}\zeta = 0, \tag{2}$$

where ξ and ζ are the scalar invariants defined by $\epsilon_{\alpha\beta}\epsilon_{\alpha\beta}$ and $\epsilon_{\alpha\sigma}\epsilon_{\sigma\beta}\epsilon_{\alpha\beta}$, respectively. In writing equation (2) we have availed ourselves of the simplicity afforded by the fact that $\epsilon_{\alpha\alpha} = 0$ (see sec. 1). *Hence the principal invariants* λ_1, λ_2, *and* λ_3 *are algebraic functions of the scalar invariants* ξ *and* ζ.

By the process of transforming the component of any scalar invariant F of the tensor ϵ to canonical co-ordinates and evaluating at the origin of this system, it is seen that F can be expressed as a function of the principal invariants λ. Hence F is a function of the invariants ξ and ζ by the above italicized statement.

Define the invariant I_n as the combination $\lambda_1^n + \lambda_2^n + \lambda_3^n$ for $n \geqq 2$. Now by the use of canonical co-ordinates it is readily seen that we have relations of the type

$$\left.\begin{aligned} I_2 &= \xi = \epsilon_{\alpha\beta}\epsilon_{\alpha\beta}; \qquad I_3 = \zeta = \epsilon_{\alpha\sigma}\epsilon_{\sigma\beta}\epsilon_{\alpha\beta}; \qquad I_4 = \epsilon_{\alpha\sigma}\epsilon_{\sigma\tau}\epsilon_{\tau\beta}\epsilon_{\alpha\beta}, \\ I_5 &= \epsilon_{\alpha\sigma}\epsilon_{\sigma\tau}\epsilon_{\tau\mu}\epsilon_{\mu\beta}\epsilon_{\alpha\beta}; \qquad I_6 = \epsilon_{\alpha\sigma}\epsilon_{\sigma\tau}\epsilon_{\tau\mu}\epsilon_{\mu\nu}\epsilon_{\nu\beta}\epsilon_{\alpha\beta}. \end{aligned}\right\} \tag{3}$$

Each invariant I_n is a function of the two invariants ξ and ζ by the above result. Calculation of these functions for the invariants I_4, I_5, and I_6, which will enter in the following discussion, leads to the relations[2]

$$I_4 = \frac{\xi^2}{2}; \qquad I_5 = \frac{5\xi\zeta}{6}; \qquad I_6 = \frac{\xi^3}{4} + \frac{\zeta^2}{3}. \tag{4}$$

3. *General Form of the Stress-Strain Relations.*—It can be shown that any tensor s, symmetric and of the second order, which is an invariant of the tensor ϵ under the group of proper orthogonal transformations, is expressible by relations of the form[3]

$$s_{\alpha\beta} = F\delta_{\alpha\beta} + G\epsilon_{\alpha\beta} + H\epsilon_{\alpha\sigma}\epsilon_{\sigma\beta}, \tag{5}$$

where F, G, and H are scalar invariants of the tensor ϵ. Hence a relation (5) exists between the stress deviation s and the rate of strain tensor ϵ, by Assumption I. Now $s_{\alpha\alpha} = 0$ and $\epsilon_{\alpha\alpha} = 0$. We can therefore eliminate the invariant F from (5) to obtain

$$s_{\alpha\beta} = G\epsilon_{\alpha\beta} + H(\epsilon_{\alpha\sigma}\epsilon_{\sigma\beta} - \frac{1}{3}\epsilon_{ij}\epsilon_{ij}\delta_{\alpha\beta}). \tag{6}$$

But the scalar invariants G and H are functions of the invariants ξ and ζ by the result in section 2. *Hence the most general relationship between the tensors* s *and* ϵ *is given by* (6), *in which* G *and* H *are functions of the scalars* ξ *and* ζ.

4. *Proportionality of the* s *and* ϵ *Tensors.*—Assume that relations (6) are satisfied with $H = 0$. Then

$$s_{\alpha\beta} = G\epsilon_{\alpha\beta}, \tag{7}$$

i.e., the tensors s and ϵ are proportional. We first consider the special case of (7) for which the invariant G is a function of ξ alone. We have

$$s_{\alpha\beta}s_{\alpha\beta} = G^2\epsilon_{\alpha\beta}\epsilon_{\alpha\beta} = G^2\xi. \tag{8}$$

Now, if $G^2\,\xi$ is not independent of ξ, equation (8) can be solved to express ξ as a uniquely determined function of the quantity $s_{\alpha\beta}s_{\alpha\beta}$ by the implicit function theorem.[4] Hence the $\epsilon_{\alpha\beta}$ are determined as functions of the variables $s_{\alpha\beta}$ by (7), and a (1, 1) correspondence between the components of the s and ϵ tensors is thus established, in contradiction to Assumption II. To avoid this situation we must have

$G^2\xi = \text{const.}$ Hence we can write $s_{\alpha\beta}s_{\alpha\beta} = 2k^2$, where k is a constant in the sense that it is independent of the variables ξ and ζ but may depend on other factors associated with the present or past state of the medium (see sec. 1). This gives the value $G = \sqrt{2}k/\sqrt{\xi}$ for the scalar invariant G in (7). In other words, we are led to the quadratic yield condition and the stress-strain relations

$$s_{\alpha\beta} = \frac{\sqrt{2}k}{\sqrt{\epsilon_{ij}\epsilon_{ij}}}\,\epsilon_{\alpha\beta}, \tag{9}$$

used by von Mises in his theory of plasticity.

We now consider the general case of the proportionality relation (7). Put $\xi_* = s_{\alpha\beta}s_{\alpha\beta}$ and $\zeta_* = s_{\alpha\sigma}s_{\sigma\beta}s_{\alpha\beta}$, corresponding to the definition of the scalars ξ and ζ in section 2. Then, from (7), we have

$$\xi_* = G^2(\xi, \zeta)\xi; \qquad \zeta_* = G^3(\xi, \zeta)\zeta. \tag{10}$$

If the functional determinant Δ of the right-hand members of equations (10) does not vanish, these equations have a solution $\xi(\xi_*, \zeta_*)$, $\zeta(\xi_*, \zeta_*)$, and hence G becomes a function of ξ_* and ζ_*.[4] Hence equations (7) admit a solution $\epsilon_{\alpha\beta}(s)$ which contradicts Assumption II. We must therefore have $\Delta = 0$ to avoid this contradiction.

Equating Δ to zero, we find

$$2\xi\frac{\partial G}{\partial \xi} + 3\zeta\frac{\partial G}{\partial \zeta} + G = 0. \tag{11}$$

To solve equation (11), we first make the transformation $\xi = e^{2x}$, $\zeta = e^{3y}$ of the variables ξ, ζ and follow this by the transformation $x = \alpha$, $y = \alpha - \beta$. Then (11) becomes $\partial G/\partial\alpha + G = 0$, and this equation has the general solution $G = A(\beta)e^{-\alpha}$, where $A(\beta)$ is an arbitrary differentiable function of β. Returning to the original variables ξ and ζ, we thus find that the general solution of (11) is given by

$$G = \frac{E(\sqrt[3]{\epsilon_{ik}\epsilon_{kj}\epsilon_{ij}}/\sqrt{\epsilon_{ij}\epsilon_{ij}})}{\sqrt{\epsilon_{ij}\epsilon_{ij}}}, \tag{12}$$

where E is an arbitrary differentiable function of the ratio $\sqrt[3]{\zeta}/\sqrt{\xi}$.

5. *The Yield Condition.*—Consider the stress-strain relations (7), with G given by (12). From (7) we deduce the two equations (10), which can be combined to give $\zeta_*^{1/3}/\xi_*^{1/2} = \zeta^{1/3}/\xi^{1/2}$. Hence, replacing the argument $\zeta^{1/3}/\xi^{1/2}$ of E by the ratio $\zeta_*^{1/3}/\xi_*^{1/2}$, it follows from (7) and (12) that

$$s_{\alpha\beta} = \frac{E(\sqrt[3]{s_{ik}s_{kj}s_{ij}}/\sqrt{s_{ij}s_{ij}})}{\sqrt{\epsilon_{ij}\epsilon_{ij}}}\,\epsilon_{\alpha\beta}. \tag{13}$$

Now multiply each member of (13) by itself and sum on the repeated indices α and β. We thus obtain the yield condition

$$s_{\alpha\beta}s_{\alpha\beta} = \left[E\left(\frac{\sqrt[3]{s_{ik}s_{kj}s_{ij}}}{\sqrt{s_{ij}s_{ij}}}\right)\right]^2. \tag{14}$$

The following result has now been proved: *When the tensors s and ϵ are propor-*

tional, the yield condition may be taken to have the form (14), in which E is any differentiable function of the ratio $\zeta_*^{1/2}/\xi_*^{1/3}$; after selection of the function E, the proportionality factor G in (7) *will be given by* (12). In particular, if we take $E = \sqrt{2}k$, where k is a material constant, we obtain from (14) the von Mises, or quadratic, yield condition, while (7) and (12) give (9) as the associated stress-strain relations.

6. *Discussion of the General Case.*—Consider the general stress-strain relations (6), in which G and H are functions of the scalar invariants ξ and ζ. The formal construction of the equations corresponding to (10) will initially involve homogeneous scalar invariants formed from the components $\epsilon_{\alpha\beta}$. These homogeneous invariants can be identified with the invariants ξ, ζ, I_4, I_5, and I_6 by (3), and the invariants I_4, I_5, and I_6 can then be eliminated by means of (4). We thus obtain

$$\left.\begin{aligned} \xi_* &= \xi G^2 + 2\zeta GH + \frac{1}{6}\xi^2 H^2, \\ \zeta_* &= \zeta G^3 + \frac{1}{2}\xi^2 G^2 H + \frac{1}{2}\xi\zeta GH^2 + \frac{1}{3}\left(\zeta^2 - \frac{1}{12}\xi^3\right) H^3. \end{aligned}\right\} \tag{15}$$

The procedure is now analogous to that employed in section 5. We first observe that the functional determinant Δ of the right-hand members of (15) must vanish in consequence of Assumption II. This gives a partial differential equation $\Delta = 0$ for the determination of G and H as functions of the variables ξ and ζ. Corresponding to any solution $G(\xi, \zeta)$ and $H(\xi, \zeta)$ of this differential equation, the quantities ξ_* and ζ_* given by (15) will be functionally dependent. *The equation expressing this functional dependence will be the yield condition associated with the stress-strain relations* (6), *in which G and H are the functions obtained as the solution of the above differential equation.*

7. *Formulation of Results in Terms of Principal Invariants.*—Denote by λ_1^*, λ_2^*, and λ_3^* the principal invariants of the stress deviation s, and, for definiteness, let us suppose that these invariants are labeled so that the inequalities $\lambda_1^* \geqq \lambda_2^* \geqq \lambda_3^*$ are satisfied. In the formulation of results in terms of principal invariants, we shall limit our attention to the proportional case (7), for which we have the simple relationship $\lambda_i^* = G\lambda_i$ $(i = 1, 2, 3)$ between the invariants λ^* and the principal invariants λ of the rate of strain tensor ϵ. Making the customary assumption that $G > 0$, the above inequalities on the λ^* imply the inequalities $\lambda_1 \geqq \lambda_2 \geqq \lambda_3$. Instead of the two variables ξ and ζ used in the above discussion, we can now select two of the three principal invariants λ. Choose these to be the invariants λ_1 and λ_3. The intermediate invariant λ_2 in the above inequalities is then determined by λ_1 and λ_3 on account of the relation $\lambda_1 + \lambda_2 + \lambda_3 = 0$, which expresses the condition that ϵ is a deviation tensor.

It was seen in section 2 that the component F of any scalar invariant of the ϵ tensor can be expressed as a function of the principal invariants λ. Hence F can be expressed as a function of the two invariants λ_1 and λ_3 by elimination of the intermediate invariant λ_2. Hence the coefficient G in (7) can be regarded as a function of the invariants λ_1 and λ_3.

We now consider the two relations

$$\lambda_1^* = G\lambda_1: \qquad \lambda_3^* = G\lambda_3. \tag{16}$$

From the preceding discussion we can immediately state that Assumption II requires the vanishing of the functional determinant

$$\begin{vmatrix} \left(\frac{\partial G}{\partial \lambda_1}\lambda_1 + G\right) & \frac{\partial G}{\partial \lambda_3}\lambda_1 \\ \frac{\partial G}{\partial \lambda_1}\lambda_3 & \left(\frac{\partial G}{\partial \lambda_3}\lambda_3 + G\right) \end{vmatrix}.$$

We thus obtain the simple equation

$$\lambda_1 \frac{\partial G}{\partial \lambda_1} + \lambda_3 \frac{\partial G}{\partial \lambda_3} + G = 0 \tag{17}$$

for the determination of G as a function of the variables λ_1 and λ_3. Now it is readily seen (cf. sec. 4) that the general solution of (17) is given by $G = A(\lambda_3/\lambda_1)/\lambda_1$, where A is an arbitrary differentiable function of the ratio λ_3/λ_1. But from (16) we have $\lambda_3/\lambda_1 = \lambda_3^*/\lambda_1^*$. Hence, from the first equation (16), we get $\lambda_1^* = A(\lambda_3^*/\lambda_1^*)$ as the yield condition. Putting

$$B\left(\frac{\lambda_3}{\lambda_1}\right) = \left(1 - \frac{\lambda_3}{\lambda_1}\right) A\left(\frac{\lambda_3}{\lambda_1}\right),$$

we obtain a modified form of the yield condition and associated stress-strain relations as follows:

$$\textit{yield condition:} \qquad \lambda_1^* - \lambda_3^* = B\left(\frac{\lambda_3^*}{\lambda_1^*}\right), \tag{18}$$

$$\textit{stress-strain relations:} \qquad s_{\alpha\beta} = \left(\frac{B(\lambda_3/\lambda_1)}{\lambda_1 - \lambda_3}\right) \epsilon_{\alpha\beta}. \tag{19}$$

Hence we can choose $B(\lambda_3^*/\lambda_1^*)$ *to be an arbitrary differentiable function of the ratio* λ_3^*/λ_1^* *in the yield condition* (18), *and after the choice of the function* B *has been made, the associated stress-strain relations will be given by* (19). If we take, in particular, $B = 2k$, where k is a material constant, we thus obtain

$$\lambda_1^* - \lambda_3^* = 2k; \qquad s_{\alpha\beta} = \frac{2k\epsilon_{\alpha\beta}}{\lambda_1 - \lambda_3}.$$

The first of these equations is the Tresca yield condition, and the second set constitutes the corresponding stress-strain relations.

[1] If the assumption of incompressibility were not made, we would define ϵ as the deviation of the rate of strain tensor. This would necessitate no modification of the discussion or results of this note.

[2] From eq. (2) we have $\lambda_i^4 = \xi\lambda_i^2/2 + \zeta\lambda_i/3$ for $i = 1, 2, 3$. Taking $i = 1, 2, 3$ successively and adding corresponding members of these equations, we immediately obtain the first relation (4). The second and third relations (4) can be deduced in a similar manner.

[3] See R. S. Rivlin, and J. L. Ericksen, "Stress-Deformation Relations for Isotropic Materials," to appear in the *Jour. Rat. Mech. Anal.*

[4] It is assumed that all invariants under consideration are continuous and have continuous first partial derivatives with respect to their arguments. This permits the construction of functional determinants and the use of the implicit function theorem.

A GENERALIZATION OF THE FRÉCHET DISTANCE OF TWO CURVES

By A. J. Ward*

INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY

Communicated by Marston Morse, April 9, 1954

Let U be a separated uniform structure on a space E, and h any curve in E. It is easy to define[1] a uniform structure $\tilde{U}$ on the space $h(E)$ whose elements are the curves in E. Again, if $f(x, y)$ is any écart (or quasi-distance) in E, there is an obvious definition of a function $D(h_1, h_2)$ analogous to the Fréchet distance of curves h_1, h_2 in a metric space, and if U is defined by a family of écarts, we show how to define $\tilde{U}$ by a suitable family of derived écarts of the type $D(h_1, h_2)$.

Morse[2] showed that for curves in a metric space the Fréchet distance is zero if and only if the curves coincide (subject to a slight restriction on the allowable parametrizations). By an extension of his argument, we show that there exists a structure $\tilde{U}_2$, defined by a subfamily of the family of écarts which defines $\tilde{U}$, which is topologically equivalent to $\tilde{U}$, and also obtain the result, analogous to that of Morse, that the structure $\tilde{U}$ is separated. Finally, we give an example to show that the structures $\tilde{U}$, $\tilde{U}_2$ are not in general the same.

The notation of this paper is chosen to agree, as far as possible, with that of a subsequent paper discussing a different and more widely applicable generalization of the Fréchet distance.

1. *Notation: The Structure $\tilde{U}$ and the Fréchet Écarts.*—Let T(T^*, T_0, etc.) denote in general any ordinally complete simply ordered set, so that T is compact, but not in general connected, in the natural order topology.[3] Throughout the present discussion, T will always be the real interval $0 \leq x \leq 1$. Let E be a space (fixed throughout) with a separated uniform structure U, and the induced topology, defined by a fundamental system of sets $V_j (j \in J)$ of pairs (x, y) of elements; the sets V_j (known as "entourages" or *vicinities*) may without loss of generality be supposed to be symmetric and open in $E \times E$, in addition to satisfying the conditions given by Bourbaki (*op. cit.*, pp. 86–87).

By a *generalized curve representation of base T*, we mean a continuous mapping $x = p(t)$, $T \to E$ (T having the order topology), such that T contains no nondegenerate interval on which p is constant. Two representations p: $T \to E$ and p^*: $T \to E$ are called *equivalent* if there is a homeomorphism v of T onto itself such that $p^* \circ v = p$. By a *generalized curve h*, we mean a maximal set of equivalent representations. When, as here, T is a closed real interval, we omit the word "generalized."

Let V_j be one of the vicinities defining the structure of E, and let $\tilde{V}_j$ be the set of ordered pairs of curves h, k in E, such that there exist representations $x = p(h, t)$ of h, $x = p(k, t)$ of k, with the property that $[p(h, t); p(k, t)]$ is in V_j for each value of t in T. It is easily seen that the family of sets $\tilde{V}_j$ possesses the properties necessary to define a uniform structure $\tilde{U}$ (possibly nonseparated) on the space of all curves in E and that, if (W_i) is an alternative fundamental set of vicinities defining the same structure U on E, then the family $(\tilde{W}_i)$ defines the same uniform structure $\tilde{U}$ as does the family $(\tilde{V}_j)$.

Suppose, now, that the structure U in E is given by a family of écarts f_i, $i \in I$. Without loss of generality, we may suppose that $0 \leq f_i(x, y) \leq 1$, all x, y, i. If (i_r), $r = 1, 2, \ldots, n$ is any finite set of indices i in I, we write $f_{(i_r)}(x, y)$ or $f_{i_1, i_2, \ldots, i_n}(x, y)$ for $\max_{1 \leq r \leq n} f_{i_r}(x, y)$; then a fundamental set of vicinities for U is given by the sets $V[(i_r), a] = V[i_1, i_2, \ldots, i_n; a]$ defined as $[(x, y) \mid f_{(i_r)}(x, y) < a]$, where a is any positive number and the finite set of indices (i_r) is also quite arbitrary. We remark that each $f_{(i_r)}(x, y)$ is necessarily a continuous function of (x, y) relative to the induced topology in $E \times E$.

We define, for each pair of curves h, k and each finite set (i_r), the *Fréchet écart* $D_{(i_r)}(h, k) = D_{i_1, i_2, \ldots, i_n}(h, k)$ as the lower bound of numbers $a > 0$ such that (h, k) is in $\bar{V}[(i_r), a]$. It is clear that it is in fact an écart and that the family of these écarts defines the structure $\bar{U}$.

We remark that it is easy to construct an example of two curves, in a space with two écarts f_1, f_2, for which $D_1(h, k) = D_2(h, k) = 0$ but $D_{1,2}(h, k) > 0$.

2. *The μ_i-Length of a Curve.*—Following Morse,[2] if h is a curve with a representation $x = p(t)$, and $f_i(x, y)$ any écart, we consider, for any n, an arbitrary set T_n of values $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$ in T, and define

$$m_{n,i}(h) = \sup_{\text{all } T_n} [\min_{1 \leq t \leq n-1} f_i\{p(t_r), p(t_{r+1})\}],$$

and

$$\mu_{h,i} = \sum_{n=1}^{\infty} 2^{-n} m_{n+1,i}(h).$$

It is clear that $\mu_{h,i}$ is independent of the representation chosen. *For a given representation* we also write $\mu_{h,i}(t)$ for the corresponding value when we add the restriction $t_n \leq t$. We show, almost exactly as in Morse's treatment, that $\mu_{h,i}(t)$ is a continuous function of t, that it is increasing in the wide sense, and that it is constant on an interval $t_1 \leq t \leq t_2$ if and only if $f_i\{p(t), p(t')\} = 0$ whenever t and t' both lie in $\langle t_1, t_2 \rangle$.

Let $\mu = (\mu_i)$ denote a point of the product-space R^I, endowed with the uniform structure induced by the metric structure of R. The mapping $\mu = \mu_h(t)$ defined by $\mu_i = \mu_{h,i}(t)$, each i, is a continuous mapping of T into R^I. The natural partial ordering of R^I induces an ordering of the set $\mu_h(T)$. Since T contains no nondegenerate interval of constancy of $p(t)$ and since, U being a separated structure, at least one $f_i(x, y)$ is positive if $x \neq y$, we see that, if $t_1 < t_2$, then $\mu_{h,i}(t_1) \leq \mu_{h,i}(t_2)$, all i, while $\mu_{h,i}(t_1) < \mu_{h,i}(t_2)$ for at least one i, so that, with the natural induced ordering, $\mu_h(t_1) < \mu_h(t_2)$. Since T is compact and $\mu_h(t)$ is continuous and 1-1, it follows that μ_h is an order-preserving homeomorphism of T onto $\mu_h(T) \subset R^I$, and the order topology of $\mu_h(T)$ coincides with its topology as a subset of R^I. Again, if $\mu_h^*(t)$ is defined for the same curve h in terms of another representation $x = p^*(t)$, where $p^* \circ v = p$, we clearly have $\mu_h^*[v(t)] = \mu_h(t)$; hence we can define uniquely a function $x = q(h, \mu)$, for μ lying in $\mu_h(T)$, by the relation $q[h, \mu_h(t)] = p(t)$.

THEOREM 1.[4] *The function $x = q(h, \mu)$ is continuous in h relative to the $\bar{U}$ topology and uniformly continuous in μ along any given curve h_0; in fact, given h_0 and given $\epsilon > 0$, $\exists\, \delta_i > 0$, such that, whenever $D_i(h_0, h) < \delta_i$ and $|\mu_i - \mu_i^0| < \delta_i$, then*

$$f_i[q(h_0, \mu^0), q(h, \mu)] < \epsilon.$$

Here δ_t depends on h_0 but not on μ_0; it is understood that μ, μ^0 lie in $\mu_h(T)$, $\mu_{h_0}(T)$, respectively.

By the continuity of $\mu_{h_0,\ i}(t)$, the compactness of T, and the fact that $\mu_{h_0,\ i}$ is constant for $t_1 \leq t \leq t_2$ only if the f_i-diameter of the set $\{p(t) \mid t_1 \leq t \leq t_2\}$ is zero, it is easily shown that $\exists \delta_i > 0$, such that

$$f_i[q(h_0, \mu^0), q(h_0, \mu')] < {}^1/_2\,\epsilon \text{ whenever } |\mu_t{}^0 - \mu_t'| < 3\delta_i$$

and the left-hand side is defined. By Morse's Theorem 27.1,[5] which applies to μ_i-length and the D_i-écart, given a point $q(h, \mu)$ on a curve h such that $D_i(h_0, h) < \delta_i$, there is a corresponding point $q(h_0, \mu')$ on h_0 such that $|\mu_i - \mu_i'| \leq 2\delta_i$, and also $f_i[q(h_0, \mu'), q(h, \mu)] < \delta_i$. Hence, if we choose $\delta_i < {}^1/_2\epsilon$ and if $|\mu_i - \mu_i{}^0| < \delta_i$, then $|\mu_i' - \mu_i{}^0| < 3\delta_i$, and the result follows.

3. *The Reduced Structure $\bar{U}_2$.*—We denote by $\bar{U}_m$ the structure defined by the family of écarts $D_{(i_r)}(h, k)$, where (i_r) is a set of not more than m indices. It is clear that, if $m \leq n$, $\bar{U}_m$ is coarser (in the wide sense) than $\bar{U}_n$, which in turn is coarser than $\bar{U}$.

LEMMA. *Given h_0, i, j, and k, and $\epsilon > 0$, there exists $\delta > 0$ such that $D_{i,\ j,\ k}(h_0, h) < \epsilon$ whenever*

$$D_{j,\ k}(h_0, h) + D_{k,\ i}(h_0, h) + D_{i,\ j}(h_0, h) < \delta.$$

We define δ_i, δ_j, and δ_k as in Theorem 1 and write $\delta = {}^1/_2 \min[\delta_i, \delta_j, \delta_k]$.

Let $x = p_0(t)$, $x = p(t)$ be representations with bases T_0, T of h_0 and h, respectively, where h satisfies the condition of the lemma. There exist homeomorphisms $v_{jk}(t)$, $v_{ki}(t)$, and $v_{ij}(t)$ of T_0 onto T, such that

$$f_{\alpha,\ \beta}[p_0(t), p \circ v_{\alpha\beta}(t)] < \delta \text{ for all } t,$$

where $\alpha, \beta = j, k$ or k, i or i, j. Define $v(t)$ as the middle one, in the natural order on T, of $v_{jk}(t)$, $v_{ki}(t)$, and $v_{ij}(t)$. We say that $f_{i,\ j,\ k}[p_0(t), p \circ v(t)] < \epsilon$, all t. Since v is clearly a homeomorphism, this will prove the lemma.

For, by applying Theorem 27.1[5] to the parts of h_0, h preceding t, $v_{\alpha\beta}(t)$ (inclusive), respectively, we have

$$|\mu_{h_0,\ \alpha}(t) - \mu_{h,\ \alpha}[v_{\alpha\beta}(t)]| \leq 2\delta \leq \delta_\alpha,$$

with a similar inequality where β replaces α. Now, given t, there is at least one choice of α, β with $i = \alpha$ or $i = \beta$ and $v_{\alpha\beta}(t) \leq v(t)$ so that $\mu_{h,\ i}[v_{\alpha\beta}(t)] \leq \mu_{h,\ i}[v(t)]$ and, similarly, at least one choice with the inequalities reversed. Thus we conclude that $|\mu_{h,\ i}[v(t)] - \mu_{h_0,\ i}(t)| \leq \delta_i$. Similar inequalities hold for j, k: by the definition of δ_i, δ_j, and δ_k [since $D_i(h_0, h) \leq D_{i,\ j}(h_0, h) < \delta_i$, and similarly for j, k] we have the required result, on applying Theorem 1.

4. THEOREM 2. *The structure $\bar{U}_2$ is topologically equivalent to the structure $\bar{U}$.*—Consider any écart $f_{i_1,\ i_2,\ \ldots\ i_n}$. By applying the lemma to the écarts $g_i = f_{i_1,\ i_2,\ \ldots,\ i_{n-}}$, $g_j = f_{i_{n-1}}$ and $g_k = f_n$, we see that, given h_0 and given $\epsilon > 0$, $\exists \delta > 0$, such that $D_{i_1,\ i_2,\ \ldots,\ i_n}(h_0, h) < \epsilon$ whenever $D_{i_1,\ i_2,\ \ldots,\ i_{n-2},\ i_{n-1}}(h_0, h)$, $D_{i_1,\ i_2,\ \ldots i_{n-2}\ i_n}(h_0, h)$, and $D_{i_{n-1},\ i_n}(h_0, h)$ are all $< \delta$. By induction, $\exists \eta > 0$, such that this holds whenever all the inequalities $D_{\alpha,\beta}(h_0, h) < \eta$, α and β being selected arbitrarily from $i_1, i_2, \ldots, i_n$, are satisfied.

Any fundamental neighborhood of h_0 in the $\bar{U}$ topology may be defined by a

finite number of inequalities of the type considered and therefore contains a neighborhood of h_0 in the $\bar{U}_2$ topology; the converse is obvious. The topologies induced by $\bar{U}$, $\bar{U}_2$ are therefore equivalent.

THEOREM 3. *$\bar{U}_2$ (and hence also $\bar{U}$) is a separated uniform structure.*

Suppose that $D_{i,j}(h_0, h) = 0$ for all i, j in I; we have to show that $h_0 = h$. Take representations $x = p_0(t)$ of h_0, $x = p(t)$ of h, and let $v(t)$ be a homeomorphism such that $f_{i,j}[p_0(t), p \circ v(t)] < \epsilon$, all t. Again applying Theorem 27.1,[5] we have $|\mu_{h_0,i}(t) - \mu_{h,i}[v(t)]| \leq 2\epsilon$, and similarly for j. That is, if t_1 and t_2 are such that $\mu_{h_0,i}(t_1) < \mu_{h,i}(t_2) - 2\epsilon$, then $v(t_1) < t_2$ and hence $\mu_{h_0,j}(t_1) \leq \mu_{h,j}(t_2) + 2\epsilon$. Since ϵ is arbitrary, we see that, whenever $\mu_{h_0,i}(t_1) < \mu_{h,i}(t_2)$, then also $\mu_{h_0,j}(t_1) \leq \mu_{h,j}(t_2)$; here i and j are any indices.

We now define $v_0(t_0)$ ($t_0 \in T_0$) as the upper bound of $\tau \in T$, such that $\mu_{h_0,i}(t_0) \geq \mu_{h,i}(\tau)$ for all i. By what we have just proved, if $\tau > v_0(t_0)$, then $\mu_{h_0,j}(t_0) \leq \mu_{h,j}(\tau)$ for all j. It follows by continuity that

$$\mu_{h_0,j}(t) = \mu_{h,j}[v_0(t)] \text{ for all } j \text{ and all } t \in T_0. \tag{A}$$

Since at least one $\mu_{h,j}(\tau)$ increases if τ increases, the mapping v_0 is uniquely determined by condition (A) and is clearly order-preserving; and, as a similar mapping must exist in the reverse direction, v_0 must define a homeomorphism of T_0 onto T. The sets $\mu_{h_0}(T_0)$, $\mu_h(T)$ therefore coincide: applying Theorem 1, we now see that $f_i[q(h_0, \mu), q(h, \mu)] = 0$ for all i, all μ in $\mu_h(T_0)$. Since U is a separated structure, we see that $q(h_0, \mu) = q(h, \mu)$, all μ in $\mu_h(T_0)$. That is, $p_0(t_0) = p^0 v(t_0)$, all t_0 in T_0, so that the curves h_0, h coincide.

All the preceding work has been so worded that it applies with scarcely any modification to the case when T, though not the closed real interval, is restricted to be homeomorphic to a fixed set T_0 which is connected (in the natural order topology). The theory of μ_t-length requires some modification if T is not connected, and this, together with an alternative generalization of the Fréchet distance, will be discussed in a subsequent paper.

5. We now show by an example that $\bar{U}_2$, $\bar{U}$ need not be the same uniform structure. Let N be any positive integer, and r any integer with $0 \leq r < 2N$.

For $3r \leq t \leq 3r + 1$, let $x(t) = a_r \sin \pi(t - 3r)$, $y(t) = z(t) = 0$; for $3r + 1 \leq t \leq 3r + 2$, let $y(t) = a_r \sin \pi(t - 3r - 1)$, $z(t) = x(t) = 0$; for $3r + 2 \leq t \leq 3r + 3$, let $z(t) = a_r \sin \pi(t - 3r - 2)$, $x(t) = y(t) = 0$, where $a_r = \epsilon_0 \min(r + 1, 2N - r)$ for a fixed $\epsilon_0 > 0$.

The functions $x(t)$, $y(t)$, $z(t)$, $0 \leq t \leq 6N$, define a curve h_0 in R^3. We define another curve h similarly, with $y(t)$ and $z(t)$ interchanged. If $p = (x, y, z)$, $p' = (x', y', z')$ are any two points of R^3, we introduce the three écarts $f_1(p, p') = |x - x'|$, $f_2(p, p') = |y - y'|$, $f_3(p, p') = |z - z'|$.

It is easy to see that $D_{2,3}(h_0, h) \leq 2\epsilon_0$; for any $\delta > 0$ we define $v_\delta(t)$ by giving its values at certain points as follows and making it linear between each consecutive pair of these points: $v_\delta(0) = 0$, $v_\delta(6N) = 6N$, and, for $r = 1, 2, \ldots, 2N - 2$, $v_\delta(3r) = 3r + 2$; $v_\delta(3r + 1) = 3r + 2 + \delta$; $v_\delta(3r + 2) = 3r + 3$; $v_\delta(3r + 2 + \delta) = 3r + 4$ [and $v_\delta(3r + 3) = 3r + 5$ by changing r to $r + 1$, if $r < 2N - 2$]. Since δ is arbitrarily small, the stated result follows. We see similarly that $D_{3,1}(h_0, h) = 0$, $D_{1,2}(h_0, h) = 0$. On the other hand, by considering the possible mappings of the portion of h_0 given by $3N - 3 \leq t \leq 3N + 3$, it is seen that $D_{1,2,3}(h_0, h) \geq kN\epsilon_0$,

where k is a positive absolute constant. Since, for any ϵ_0, however small, we may choose N with $kN\epsilon_0 \geq 1$, it follows that the vicinity $[h, k|D_{1,2,3}(h, k) < 1]$ in $\bar{U} = \bar{U}_3$ contains no vicinity in $\bar{U}_2$.

* The work contained in this paper was carried out while the author was in receipt of a Smith-Mundt grant tenable at the Institute for Advanced Study, Princeton, New Jersey. The author wishes to thank Professor Marston Morse for suggesting the problem and for advice on the presentation of the results.

[1] For the details see sec. 1 below; for the terminology see N. Bourbaki, *Topologie gènèrale* ("Actualités scientifiques et industrielles," No. 858 [Paris] or No. 1142 [rev. ed.]), chap. ii, and *ibid.* ("Actualités scientifiques et industrielles," No. 1045 [1948]), chap. ix, pp. 1–8.

[2] M. Morse, *Topological Methods in the Theory of Functions of a Complex Variable* ("Ann. Math. Studies," No. 15 [Princeton, N.J., 1947]), pp. 99–107.

[3] G. Birkhoff, *Lattice Theory* (2d ed.) ("Amer. Math. Soc. Colloquim Pub." [1948]), pp. 39-41.

[4] Cf. Morse, *op. cit.*, p. 105, Theorem 27.2; the proof is only slightly altered.

[5] *Ibid.*, p. 104.

ESTIMATION OF THE COMPONENTS OF STOCHASTIC STRUCTURES

BY J. WOLFOWITZ*

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY

Communicated by John von Neumann, April 19, 1954

1. The purpose of this note is to illustrate a new development of the minimum distance method by application to two important stochastic structures. A precise definition of a stochastic structure is to be found in a paper by Koopmans and Reiersol[1] but is unnecessary for comprehension of the present paper. The present method is superior for purposes of estimation to that developed by Wolfowitz.[2-4] It applies to many general structures. Linearity, normality, etc., are entirely unnecessary, although the structures chosen for illustration in the present paper are linear for simplicity and importance. Application of the method can be made routine to a considerable degree. Under reasonable conditions, often satisfied, all unknown components, including distribution functions, can be strongly consistently[5] estimated (what this means will be apparent shortly). A statement of the general results and proofs will appear elsewhere.

2. Let $u_0, u_1, u_2, \ldots$, be independent chance variables with the same distribution function (d.f.) $G(z)$, which is unknown to the statistician, as is a constant α, $|\alpha| < 1$. To avoid the trivial we assume that u_i is not constant with probability 1. Define

$$x_i = u_i + \alpha u_{i-1}, \qquad i = 1, 2, \ldots, \tag{2.1}$$

and

$$\lambda_n = (x_1, \ldots, x_n). \tag{2.2}$$

Let $F(z)$ be the (unknown to the statistician) d.f. of x_i, and $F_n(z)$ be the empiric d.f. of λ_n. It is well known that $F(z)$ determines α ($|\alpha| < 1$) uniquely. We shall assume only that $F(z)$ satisfies some conditions sufficient so that $F(z)$ determines $G(z)$ uniquely.[6] For example, it is sufficient for this purpose that the characteristic

function of $F(z)$ have no zeros. In the matter of assumptions our treatment is very different from that of Wold.[7]

Let $\delta^*(A(z), B(z))$ be any of many definitions of distance between two d.f.'s which (definitions) satisfy certain weak conditions. For definiteness we shall here assume that δ^* is the Lévy distance, defined as the infimum of all positive h such that

$$A(z - h) - h \leq B(z) \leq A(z + h) + h.$$

Let $A(z) * B(z)$ be the convolution of $A(z)$ and $B(z)$. Let $H(z)$ be any d.f., and a any number less than 1 in absolute value. Write $S = (a, H(z))$, and define

$$\begin{aligned} D(z|S) &= H(z) * H\left(\frac{z}{a}\right) && (a > 0), \\ &= H(z) * \left[1 - H\left(\frac{z}{a} + 0\right)\right] && (a < 0), \\ &= H(z) && (a = 0). \end{aligned}$$

Thus $D(z|S)$ is the d.f. of x_i when $\alpha = a$ and $G(z) = H(z)$. Define

$$m(S) = \delta^*(F_n(z),\ D(z|S)).$$

Let $S^*(\lambda_n) = (a(S^*), H(z|S^*))$ be any S whose elements are Borel measurable functions of λ_n (in the case of $H(z|S^*)$ this means that, for every z, $H(z|S^*)$ is a Borel measurable function of λ_n) such that

$$m(S^*(\lambda_n)) < \frac{1}{n^2} + \inf m(S), \tag{2.3}$$

the infimum operation being over all S. Then $a(S^*)$ and $H(z|S^*)$ are strongly consistent estimators of α and $G(z)$, respectively. This means that, with probability 1 as $n \to \infty$,

$$a(S^*) \to \alpha, \tag{2.4}$$

$$\delta^*(H(z|S^*), G(z)) \to 0, \tag{2.5}$$

identically in $G(z)$ and α.

3. An important structure with a very long history of study (see, for example, Koopmans and Reiersol,[1] Wald,[8] and Reiersol,[9] and the long list of references given there) is as follows: Let

$$x_1 = \xi + v_1, \tag{3.1}$$

$$x_2 = \alpha + \beta\xi + v_2, \tag{3.2}$$

where α and β are constants which may be any real values (including $\pm\infty$ in a sense described by Reiersol[9]), and ξ, v_1, and v_2 are chance variables. It is known to the statistician that (v_1, v_2) are jointly normally distributed with zero means, independently of ξ. The parameters α and β, the covariances $\mu_{ij} = Ev_iv_j$, and the d.f. $L(z)$ of ξ are unknown to the statistician. The problem is to give strongly consistent estimators of these as functions of

$$\lambda_n = \{(x_{1i}, x_{2i}),\quad i = 1, \ldots, n\},$$

where (x_{1i}, x_{2i}), $i = 1, 2, \ldots$, ad inf., are independent chance variables with the same d.f. $F(z_1, z_2)$, which is the d.f. of (x_1, x_2).

The problem which has been attacked longest is that of estimating α and β. A very great advance was made by Reiersol,[9] who proved that the condition that $L(z)$ be not normal is sufficient for $F(z_1, z_2)$ to determine α and β uniquely. That this condition is necessary had already been noted by Neyman.[10] Thus the problem arose of giving strongly consistent estimators of α and β under no assumptions except the indispensable one of nonnormality of $L(z)$. Most papers on the problem have made various assumptions about $L(z)$, among other things, and often these assumptions have been subject to criticism. In the remainder of this section we shall assume only that $L(z)$ is not normal.

Strongly consistent estimators of α and β under no assumptions except the nonnormality of $L(z)$ have been given recently essentially by Neyman[11] and by Wolfowitz.[2] These authors constructed (different) functions (really classes of functions) $a(\lambda_n)$ and $b(\lambda_n)$ of λ_n, such that, with probability 1 as $n \to \infty$,

$$a(\lambda_n) \to \alpha, \qquad b(\lambda_n) \to \beta,$$

identically in (nonnormal) $L(z)$, μ_{11}, μ_{12}, μ_{22}, α, and β. Wolfowitz[3] gave a strongly consistent estimator of

$$\mu_{22} - 2\beta\mu_{12} + \beta^2\mu_{11},$$

the variance of $(v_2 - \beta v_1)$.

Let S be the generic designation of a complex

$$(\alpha^*, \beta^*, \mu_{11}^*, \mu_{12}^*, \mu_{22}^*, L^*(z))$$

whose first five elements are real constants such that the matrix

$$\begin{Vmatrix} \mu_{11}^* & \mu_{12}^* \\ \mu_{12}^* & \mu_{22}^* \end{Vmatrix}$$

is nonnegative definite, and $L^*(z)$ is a d.f. Let $D(z_1, z_2 | S)$ be the d.f. of (x_1, x_2) as given by equations (3.1) and (3.2) when $\alpha = \alpha^*, \beta = \beta^*, \mu_{ij} = \mu_{ij}^*$, $i, j = 1, 2$, and $L(z) = L^*(z)$. Two complexes S_1 and S_2 such that $D(z_1, z_2 | S_1) \equiv D(z_1, z_2 | S_2)$ will be called "equivalent." It can be deduced from the results of Reiersol[9] that among all (equivalent) complexes S such that $D(z_1, z_2 | S) \equiv F(z_1, z_2)$ there is a unique one, which we shall call "canonical," which is such that μ_{11}^* and μ_{22}^* are both maximal, and $L^*(z)$ has no Gaussian component. ($L^*(z)$ has no Gaussian component if it cannot be represented as the convolution of two d.f.'s, of which one is Gaussian with positive variance.) All equivalent complexes can be easily obtained from the canonical complex and have, by a theorem of Reiersol, the same first two components.

Let $F_n(z_1, z_2)$ be the empiric d.f. of λ_n. Define, for any complex S,

$$m(S) = \delta^*(F_n(z_1, z_2), D(z_1, z_2 | S)).$$

Let $S^*(\lambda_n)$ be any complex S whose elements are Borel measurable functions of λ_n such that

$$m(S^*(\lambda_n)) < \frac{1}{n^2} + \inf m(S), \tag{3.3}$$

where the infimum operation in relation (3.3) is over all S (it is sufficient to restrict one's self to S whose $L^*(z|S)$ has no Gaussian component). Then it can be shown that, with probability 1,

$$\alpha^*(S^*(\lambda_n)) \to \alpha, \qquad \beta^*(S^*(\lambda_n)) \to \beta.$$

The minimum-distance method gets its name from relations analogous to (2.3) and (3.3.).

Let $Y_1, \ldots, Y_n$ be independent chance variables with the common d.f. $H(z)$ and the empiric d.f. $H_n(z)$. Let $d(n)$ be any positive function defined for all positive integers n such that $d(n) \to 0$ as $n \to \infty$, and

$$P\{\delta^*(H(z), H_n(z)) > d(n) \text{ for infinitely many } n\} = 0.$$

There are many such functions: it is easy to verify that $n^{-1/10}$ is such a function, but this is a crude result. If $H(z)$ is continuous and Y_i is one-dimensional, there is available the sharp result of Chung,[12] according to which the function

$$\left(\frac{\log \log n}{cn}\right)^{1/2}$$

is a function $d(n)$ if $0 < c < 2$.

One method of estimating all the components of the canonical complex Σ is the following "small-distance" method: Let $S_0(\lambda_n)$ be any complex S whose elements are Borel measurable functions of λ_n such that

$$\mu_{11}^*(S_0) + \mu_{22}^*(S_0) > -\frac{1}{n^2} + \sup\,[\mu_{11}^*(S) + \mu_{22}^*(S)],$$

where the supremum operation is over all S such that $m(S) \leq d(n)$. If no S exists such that $m(S) \leq d(n)$, let S_0 be defined in any manner subject to the requirement of measurability. Then it can be shown that, with probability 1 as $n \to \infty$,

$$\alpha^*(S_0(\lambda_n)) \to \alpha, \quad \beta^*(S_0(\lambda_n)) \to \beta,$$
$$\mu_{ij}^*(S_0(\lambda_n)) \to \mu_{ij}^\circ, \quad i, j = 1, 2,$$
$$\delta^*(L^*(z|S_0(\lambda_n)), L^\circ(z)) \to 0,$$

where

$$\Sigma = (\alpha, \beta, \mu_{11}^\circ, \mu_{12}^\circ, \mu_{22}^\circ, L^\circ(z))$$

is the canonical complex determined by $F(z_1, z_2)$. Thus we have strongly consistent estimators of every component of the structure (3.1)–(3.2).

* Research under contract with the Office of Naval Research.

[1] T. C. Koopmans and O. Reiersol, "The Identification of Structural Characteristics," *Ann. Math. Stat.*, **21**, No. 2, 165–181, 1950.

[2] J. Wolfowitz, "Consistent Estimators of the Parameters of a Linear Structural Relation," *Skand. Aktuarietidskr.*, pp. 132–151, 1952.

[3] J. Wolfowitz, "Estimation by the Minimum Distance Method," *Ann. Inst. Stat. Math.* (Japan), **5**, No. 1, 9–23, 1953.

[4] J. Wolfowitz, "Estimation by the Minimum Distance Method in Nonparametric Stochastic Difference Equations," *Ann. Math. Stat.*, **25**, No. 2, 1954.

[5] The term "strongly consistent" seems more suggestive and hence preferable to the term "superconsistent," which we employed in earlier papers.

[6] If $F(z)$ does not determine $G(z)$ uniquely, then relation (2.5) need not hold. Relation (2.4) holds whether $G(z)$ is uniquely determined or not.

[7] H. Wold, *A Study in the Analysis of Stationary Time Series* (Uppsala: Almquist & Wiksells, 1938).

[8] A. Wald, "The Fitting of Straight Lines if Both Variables Are Subject to Error," *Ann. Math. Stat.*, **11**, 284–300, 1940.

[9] O. Reiersol, "Identifiability of a Linear Relation between Variables which Are Subject to Error," *Econometrica*, **18**, No. 4, 375–389, 1950.

[10] J. Neyman, "Remarks on a Paper by E. C. Rhodes," *J. Roy. Stat. Soc.*, **100**, 50–57, 1937.

[11] J. Neyman, "Existence of Consistent Estimates of the Directional Parameter in a Linear Structural Relation between Two Variables," *Ann. Math. Stat.*, **22**, No. 4, 497–512, 1951.

[12] K. L. Chung, "An Estimate Concerning the Kolmogoroff Distribution," *Trans. Am. Math. Soc.*, **67**, No. 1, 36–50, 1949.

---

*THE TEMPERATURE-PRESSURE-INHIBITOR RELATIONS OF BACTERIAL LUMINESCENCE IN VITRO**

BY BERNARD L. STREHLER† AND FRANK H. JOHNSON

BIOLOGY DIVISION, OAK RIDGE NATIONAL LABORATORY, OAK RIDGE, TENNESSEE, AND DEPARTMENT OF BIOLOGY, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY

Communicated by E. Newton Harvey, May 21, 1954

Interrelations in the quantitative effects of temperature, hydrostatic pressure, enzyme inhibitors, and other factors affecting bacterial luminescence have been studied at considerable length in an effort to advance the understanding of fundamental mechanisms controlling reaction rates in living cells.[1–5, 6–8] Although interpretations based on kinetic analyses of data obtained through this approach have helped clarify, and in some cases have anticipated, certain phenomena observed with various other processes (e.g., protein denaturation,[9] enzyme activity, [10] respiration,[11] growth and cell division,[12, 13] specific precipitation,[14] disinfection,[12, 15] narcosis,[16] and others[17]), further and more direct evidence from studies of the same system in extracts, apart from the complicated organization of the intact cells, has been lacking. Only recently have attempts to obtain cell-free extracts of the bacterial luminescent system been unquestionably successful.[18]

The results described in the present study with the extracted system provide a basis for a more detailed analysis than was possible with intact cells. The data permit a probable identification of pressure-sensitive steps in a reaction system involving chemically defined components, viz., dihydrodiphosphopyridine nucleotide ($DPNH_2$), flavin mononucleotide (FMN) and palmitic or other long-chain aliphatic aldehydes favorable to a bright, long-lasting luminescence in vitro.[19, 20] There is evidence that these components, with the complement of enzymic proteins, comprise the major part of the light-emitting system in vivo,[19, 21] and the results of the present study are consistent with this view.

Methods.—The enzyme preparation was obtained by extracting the cells of *Achromobacter fischeri* with cold acetone, as described earlier.[19] Four to 5 per cent by weight of the dry residue was taken up in distilled water and centrifuged for 1 hour at 70,000 × gravity to remove particulate matter. The supernatant was diluted

1:10 with 0.1 *M* sodium phosphate buffer, pH 7.0. The usual reaction mixture consisted of this enzyme solution, to which were added $DPNH_2$, FMN, and decaldehyde in final concentrations of approximately 1 mg., 0.3 μg., and 7μg., respectively, per milliliter, resulting in a "saturated system" whose luminescence decayed slowly at room temperature, or more gradually at lower temperatures, as the $DPNH_2$ was consumed.

In order to measure the effects of pressure, the reaction mixture was placed in a cylindrical aluminum cuvette, capacity about 4 ml., designed so that solutions could be mixed under pressure.[22] The open end of the cylinder was sealed by waterproof cellulose acetate, the cuvette was placed in a stainless-steel pressure chamber or "bomb,"[23] and the bomb was filled with water at the desired temperature before attaching it to a hydraulic pump. The bomb was then placed in a water bath to maintain a constant temperature. Luminescence emerging through the cellophane closure of the cuvette passed through a thick Herculite window of the bomb and thence through a plate-glass wall of the water bath onto a photomultiplier tube. The whole system was carefully shielded from extraneous light. Automatic recording of the amplified output of the photomultiplier was made with an Esterline Angus instrument having a period of approximately $^1/_2$ second.

Pressure-Temperature Relations of Luminescence Intensity.—Figure 1 illustrates the changes in light intensity of the saturated system on sudden application and sudden release of 6000 psi at temperatures below, near, and above that of the optimum, about 25° C., under normal pressure.[19] The effects of a given pressure are profoundly influenced by temperature, and they are reversible, as they are in cells.[1, 2]

At all temperatures, the time courses of the changes in intensity induced by pressure have certain features in common. In Figure 1, these features are referred to by the letters on the 3° C. curve. Thus increased pressure causes a rapid increase in intensity from a steady-state level (*a*) to a transitory peak, or "spike" (*b*), followed by a much slower rate of fall (*c*) to a new steady state (*d*). The level of (*d*) is higher or lower than the original level (*a*), depending on the temperature. Release of pressure causes a rapid, transitory fall, or "dip" (*e*), followed by a much slower rate of rise (*f*) to a new steady state (*g*). The level of (*g*) corresponds to the original level (*a*), provided that allowance is made for the amount of decay during the interval of time involved. At high temperatures, allowance must also be made for the amount of thermal inactivation of the enzyme system.

Similar changes occur in the living cells, but more rapidly, possibly because of a structural organization rather than the homogeneous solution of the reactants in extracts. Figure 1 includes for comparison a curve obtained with a cell suspension of this species.[24] The difference in steady-state levels, which could be readily measured, is essentially the same in the cells and extracts. With cells, however, the spikes and dips occurred so rapidly that their true magnitudes could not be determined by the method employed. Evidence of dips has been noted earlier and has been referred to as "black-outs," with cells of *Photobacterium phosphoreum*;[2] evidence of spikes in luminescence of cells has also been observed.[17]

Interpretation of the "Spikes" and "Dips."—A bright but very rapidly decaying luminescence occurs on adding reduced FMN to the enzyme plus aldehyde, with no appreciable $DPNH_2$ present, i.e., with no $DPNH_2$ added.[21] Using the technique described earlier for mixing the reactants inside the bomb[22] and stepping up

the amplification at short intervals as the intensity of luminescence rapidly decreased, it was possible to apply and release pressure several times during the course of the decay. At 2.5° and 16° C., with the partial system of $FMNH_2$, enzyme, and aldehyde, luminescence always rose abruptly on application of pressure and dropped

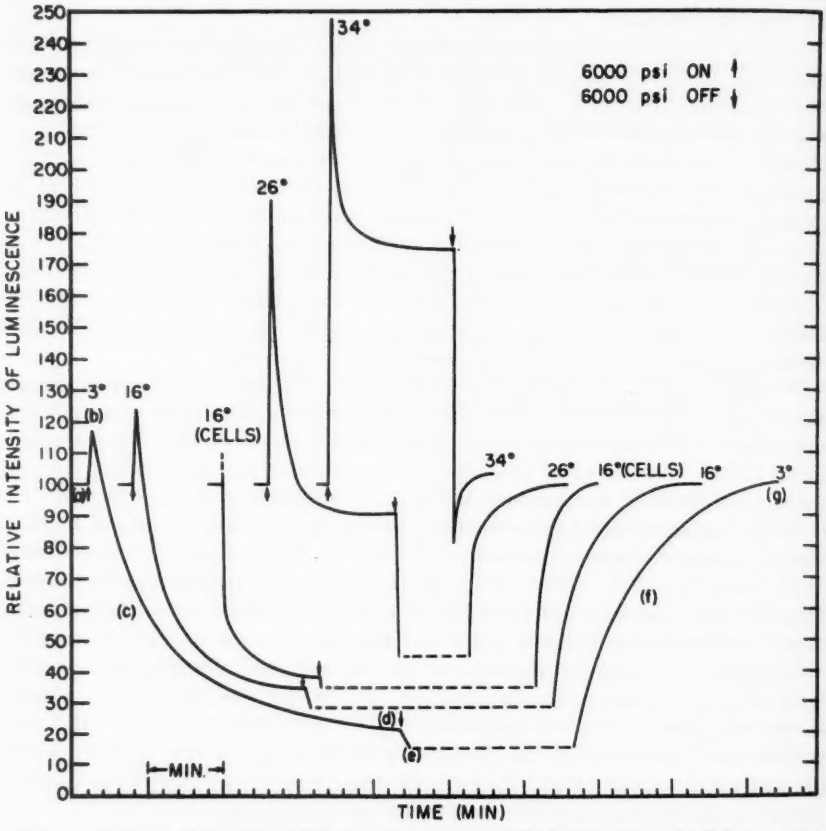


FIG. 1.—Influence of pressure at different temperatures on the luminescence of cell-free extracts of *A. fischeri* with added $DPNH_2$, FMN, and decaldehyde (saturated system). Arrows pointing upward indicate the time when pressure was applied; arrows pointing downward, when it was released. The curves for rising intensity after release of pressure (except the curve for 34° C.) have been arbitrarily displaced to the right on the abscissae, to avoid intersecting lines. The intensity at normal pressure, just before raising the pressure, is arbitrarily taken equal to 100 at each temperature and all other points computed relative to this value. Allowance was made for the amount of decay when it was significant during the periods involved, as at the higher temperatures. The increased pressure was 6000 psi at each temperature except 3° C., where it was 6500 psi. Data obtained with intact cells of *A. fischeri* at 16° C. are included for purposes of comparison.

abruptly on release of pressure. The rapidity of the decay made it difficult to determine whether or not there was a depression of luminescence under pressure corresponding to the lower steady-state level under pressure when $DPNH_2$ is present. Careful analysis of the data failed to reveal evidence of such a depression.

Although the quantitative significance of the data is limited by fairly large ex-

perimental errors associated with the rapid decay, an apparent volume change of activation,[25] $\Delta V\ddagger$, could be estimated from the magnitude of the abrupt changes in intensity due to pressure. In calculating $\Delta V\ddagger$ with the aid of the relation

$$\Delta V\ddagger = RT\,[\ln (k_{p1}) - \ln (k_{p2})]/(p_2 - p_1), \qquad (1)$$

the k's are assumed to be proportional to the intensity of luminescence at any moment. In equation (1), R is the gas constant in cubic centimeter atmospheres (= 82.07), T the absolute temperature, and p the hydrostatic pressure in atmospheres. For the change from normal pressure to 8000 psi, or for the reverse change, at either 2.5° or 16° C., the value of $\Delta V\ddagger$ was of the order of -10 cc/mole. Similar calculations, based on the spike heights and dip depths observed with the complete system containing an excess of $DPNH_2$, gave an average value, for various pressures at 3° C., of about -10 cc/mole. Under the conditions involved, the luminescent oxidation of $FMNH_2$ apparently proceeds with a small net volume increase of activation. This furnishes a plausible explanation for the spikes and dips at low temperatures. Spikes and dips represent opposite effects on the same reactions; they are merely opposite in direction, corresponding to the opposite directions of the change in pressure.

Influence of Temperature on the Spikes.—At 16° C. the spike is slightly higher than it is at 3° C. This small increase in height at 16° as compared to 3° C. is probably due to the small, sudden increase in temperature when hydrostatic pressure is suddenly applied at 16° C. The calculated change in temperature, due to the change in pressure from 1 to 1000 atm. (14,700 psi), is −0.42° at 0° C., +1.5° at 20° C., and +2.1° at 40° C.[26] There is no change at 4° C., where the coefficient of expansion of water is 0.

At temperatures close to and exceeding the optimum, there is a large increase in spike heights. The total volume change, computed with equation (1) in the manner described, is of the order of -50 cc/mole. This value is of the same order as the 50–100 cc/mole estimated earlier,[2, 5] by use of the thermodynamic equation analogous to equation (1), for the volume change of reaction, ΔV, in the reversible thermal denaturation of the luminescent enzyme system in vivo. This denaturation is accompanied by a volume increase; under pressure the equilibrium is shifted toward the undenatured form of the enzyme involved, and luminescence is thereby increased. With the extracts, qualitative observations gave evidence of a corresponding equilibrium reaction, in that brief exposures of the system to high temperatures greatly diminished the intensity of luminescence, followed by rapid recovery on cooling. The simplest interpretation of the large increase in spike height at relatively high temperatures is that pressure, as well as cooling, shifts this mobile equilibrium toward the active state of the enzyme, resulting in an immediate increase in luminescence. The same mechanism would account, in part, for the observed increases in steady-state levels under pressure at temperatures above the normal optimum (Figs. 1 and 2).

Steady-State Levels.—The percentage differences between steady-state levels as a function of pressure at different temperatures are plotted in Figure 2 from experiments similar to those illustrated in Figure 1. At the higher temperatures the points are approximate, owing to difficulties in correcting for rapidly decaying luminescence caused by both the rapid oxidative depletion of $DPNH_2$ and the

simultaneous thermal destruction of the enzyme system. Even so, it is evident that the changes in steady-state levels of luminescence are much the same in these extracts and in the psychophilic organism *P. phosphoreum*, studied earlier;[2] the original data obtained with cell suspensions of the latter have been replotted in Figure 2 for convenience of comparison. Corresponding effects of pressure occur at temperatures several degrees higher in the extracts of *A. fischeri* than in the cells

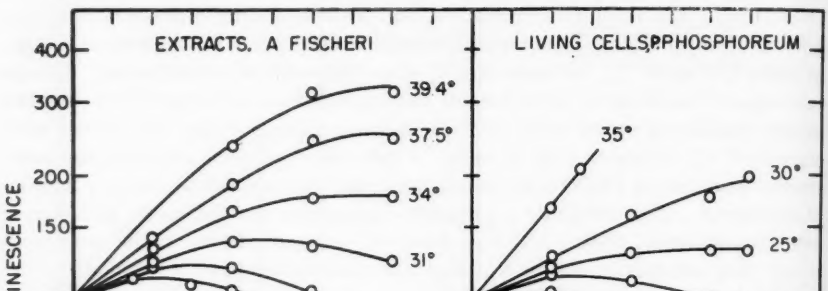


FIG. 2.—Steady-state levels of luminescence in the saturated system of *A. fischeri* extracts (*left*) and in living cells of *P. phosphoreum* (*right*). The data on the right are replotted from Brown, Johnson, and Marsland, *J. Cellular Comp. Physiol.*, **20**, 151–168, 1942. The initial intensity at normal pressure is arbitrarily taken equal to 100 at each temperature, and allowance is made for decay, when significant, in the luminescence of extracts. The lower of the two curves for 26° C. was obtained with an enzyme solution that had stood for several days at room temperature.

of *P. phosphoruem*. This temperature relation, however, is not unexpected. The normal optimal temperature for luminescence of *A. fischeri*, in either cells or extracts, is likewise several degrees higher than that in cells of *P. phosphoreum*. For reasons discussed earlier,[2] the direction and amount of the pressure effect are generally related to the temperature characteristics of the specific biological system.

In Figure 2 the slope of the line for extracts at 3° C. indicates a volume change of activation of +86 cc/mole. This apparent $\Delta V\ddagger$ is not only much larger than that

estimated for the luminescent reaction in the absence of $DPNH_2$ but is also opposite in sign. The simplest interpretation is that, under the conditions involved, a large volume increase of activation accompanies the reduction of FMN by $DPNH_2$. This is presumptive evidence for the site of the large volume increase of activation in the luminescent reaction in vivo.

At relatively high temperatures, several reactions that are sensitive to pressure become significant in determining the over-all rate, and the inhibitory effects of pressure observed at low temperatures become overridden. The simplest explanation for the increase in steady-state levels at elevated temperatures under pressure is in terms of the reversible denaturation reaction referred to above. Under pressure the steady-state level of luminescence is higher because the amount of active enzyme is greater. Thus the simplest interpretation is that the same reaction is primarily responsible for the large increase in spike heights and in steady-state levels under pressure. Values of ΔV computed from the slopes of the 37.5° and 39.4° C. curves in Figure 2, between normal pressure and 4000 psi, are 58 and 65 cc/mole, respectively. It is of interest that these values are similar to those computed from the spikes (Fig. 1, and similar experiments).

Thus, at temperatures above the normal optimum, at least three reactions potentially influence the net observed effect of pressure. These reactions include, according to the interpretations expressed, (1) reduction of FMN by $DPNH_2$, with a large volume increase of activation; (2) the luminescent oxidation of FMN, with a small volume decrease of activation; and (3) reversible denaturation of an enzyme or enzymes, with a large volume increase of reaction. Extensive data are needed in order to disentangle the several equilibria and rate processes that are significant at these temperatures and to evaluate precisely the influence of pressure on any one of them.

Rate of Change between Steady States.—With the saturated system, between 3° and 26° C., the change between steady states—i.e., in Figure 1, from the peak of the spike (*b*) to level (*d*), and from the base of the dip (*e*) back to level (*g*)—follows first-order kinetics, as illustrated in Figure 3. The rate appears to be slightly less under pressure, but the difference is small enough to be of questionable significance. Moreover, the rate does not vary significantly with the amount of pressure. The rate clearly increases, however, with temperature. The apparent activation energy for this increase in rate, between 3° and 16° C., is about 8000 cal., while between 16° and 26° C. it is about 17,000 cal.

Influence of Aldehyde.—The amount of aldehyde present profoundly affects not only the intensity of luminescence but also the response of the system to pressure. In the complete absence of added aldehyde, luminescence of the remainder of the system is less than one one-hundredth the intensity that occurs with sufficient aldehyde to cause a maximal increase, and it could not be measured accurately by the method available in this study. It was evident, however, that at 16° and 26° C., all effects of pressure are greatly reduced in the absence of added aldehyde. At 26° C. the spikes are greatly reduced, and the steady-state levels under pressure are only slightly affected. It may be significant that, at this temperature, steady-state levels are only slightly affected by pressure even with maximal aldehyde (Fig. 2). Without aldehyde at 16° C., the spikes could not be clearly determined, but no pronounced depression occurred in the steady-state level under pressure.

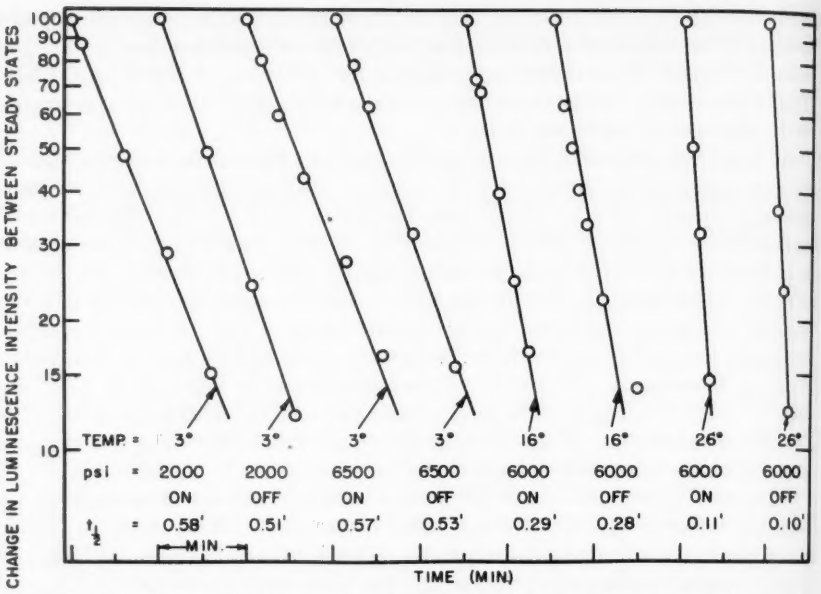


FIG. 3.—Rate of change in intensity of luminescence in the saturated system following application (*on*) and release (*off*) of pressure at different temperatures. The half-time ($t_{1/2}$) for the change between steady states is indicated below the respective lines. The lines, plotted relative to an arbitrary value of 100 at $t = 0$, represent the progressive difference in intensity, with time, between the peak of the spike and steady-state level under pressure, and the progressive difference, with time, between the lowest point of the dip and the steady-state level after pressure, respectively.

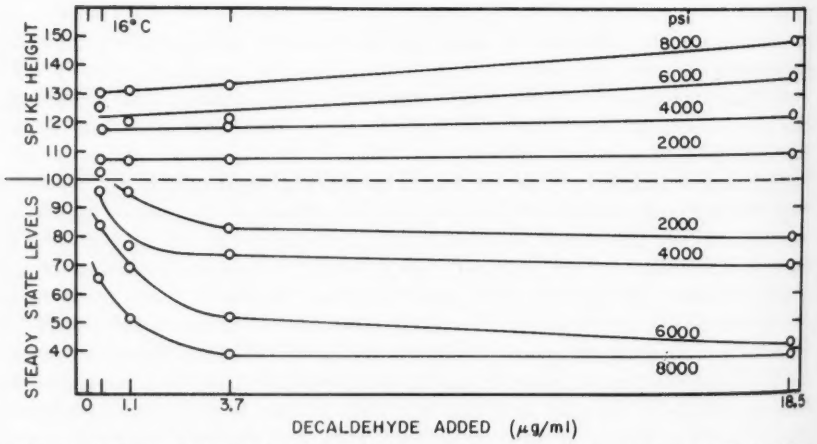


FIG. 4.—Influence of decaldehyde concentration on the spikes and steady-state levels at different pressures, at 16° C. With no decaldehyde added, the intensity of luminescence was too low to measure accurately by the method employed.

Submaximal concentrations of aldehyde, such as 0.37 μg. per milliliter, result in practically normal spike heights but in a reduced effect of pressure on the steady-state levels (Fig. 4). At this concentration, the kinetics of change between steady states is more complicated than with maximal concentrations (see below). Thus, at 16° C. the spikes are sharper, with a very rapid initial drop from the peak, followed by a relatively slow change to a lower steady state under pressure.

Action of Sulfanilamide, Alcohol, and Urethan.—Qualitative observations showed that 0.02–0.01 *M* sulfanilamide greatly inhibits the steady-state luminescence of extracts at low temperatures, but the inhibition is markedly decreased as the temperature is raised. The reverse effect of temperature was observed with 1.25 *M* ethyl alcohol, which causes a relatively slight inhibition at low temperatures.

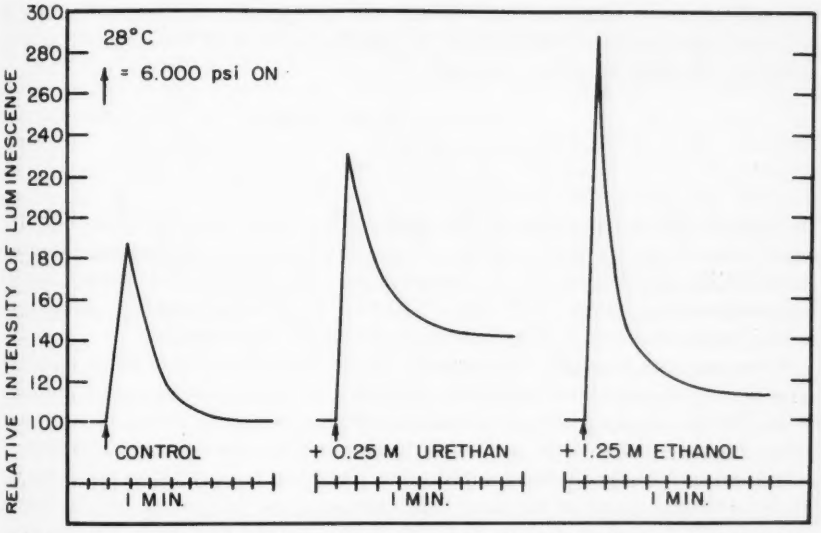


FIG. 5.—Change in intensity of the saturated system following application of 6000 psi at 28° C., in a control and in aliquot portions containing inhibitory concentrations of ethyl alcohol and urethan, respectively. The initial intensity before pressure is arbitrarily taken equal to 100 in each case, and no allowance is made for decay.

The inhibition increases in extent as the temperature is raised. Similar effects of temperature on the inhibition of luminescence by sulfanilamide and alcohol, respectively, occur in cells.[4, 7, 8] The interpretation has been discussed at length.[8, 17] Sulfanilamide acts as if it combines with or in place of a prosthetic group of an enzyme, independently of the reversible thermal denaturation, whereas drugs such as alcohol or urethan catalyze the reversible, as well as the irreversible, denaturation of the protein moiety. Since the denaturation reactions proceed with large volume increases, the effect of pressure is to reverse or to retard them, respectively, thereby reducing the inhibitory effects of the drugs that catalyze them.

Quantitative data, illustrated in Figure 5, show that, in the presence of moderately inhibitory concentrations of alcohol or urethan, pressure increases both the spike and the steady-state levels of luminescence of extracts, as compared to a

control without these drugs. There was no appreciable effect of pressure on the inhibition by 0.015 *M* sulfanilamide under similar conditions. These effects of pressure are similar to those previously observed[27] with cells by a method which could not measure the spikes.[3, 8, 17]

The concentration of alcohol or urethan required for a moderate inhibition of luminescence in extracts is roughly twice that required in cell suspensions of this species; the required concentration of sulfanilamide is of the order of 100 times greater in the extracts than in the cell suspensions. Subinhibitory concentrations of alcohol cause a pronounced increase in luminescence of extracts. Under certain conditions, urethan and alcohol increase the luminescence of cells also.[7, 8, 17]

Discussion.—Interpretation of the foregoing observations on the saturated system may be aided by a simplified diagram of the essential reactions, as follows. The role of the enzymes, and also of reactions such as thermal denaturations and equilibria with inhibitors which affect the catalytic activity of the enzymes, as well as the role of aldehyde, are here omitted.

$$A \xrightarrow{k_1} B \begin{cases} \xrightarrow{k_2} C + \text{light} \\ \xrightarrow{k_3} C'. \end{cases} \qquad (2)$$

In reactions (2), A represents $DPNH_2$ plus FMN, B represents DPN plus $FMNH_2$, while C and C' are reaction products. The rate constant k_1 is for hydrogen transfer from $DPNH_2$ to FMN, k_2 for the luminescent oxidation of $FMNH_2$, and k_3 for all nonluminescent reactions of $FMNH_2$. The intensity of luminescence, I, at any moment is proportional to $k_2[B]$, the brackets denoting concentration.

When pressure is applied or released, the numerical values of the k's change immediately, in accordance with their respective $\Delta V\ddagger$'s and the amount of pressure. With $DPNH_2$ in excess, its concentration remains essentially constant over short periods of time. The "pool size" of $FMNH_2$ remains constant under steady-state conditions but changes from an initial value $[B_0]$ when the k's change under the influence of temperature or pressure. An expression can be derived for the rate of change to a new steady state, in line with the theoretical treatment of consecutive reactions given elsewhere for the effects of pressure on the rate of ciliary beating.[17] The following relation, in which b is a proportionality constant, is obtained:

$$I = bk_2[B] = bk_2 \left\{ [B_0] - \frac{k_1[A]}{k_2 + k_3} \right\} e^{-(k_2 + k_3)t} + \frac{k_1[A]}{k_2 + k_3}. \qquad (3)$$

The rate of change between steady states is given by the exponential factor. Thus the slopes of the lines in Figure 3 are proportional to $(k_2 + k_3)$. Since k_2 is sensitive to pressure, as shown by the spikes and dips (Fig. 1), whereas $(k_2 + k_3)$ is not significantly affected by pressure (Fig. 3), it follows that either (1) k_3 is very much larger than k_2, i.e., by far the greatest fraction of $[B]$ disappears by nonluminescent reactions, or (2) compensatory changes in k_2 and k_3 take place under pressure.

With reference to the meaning of the "$^1/_2$ rise time," when an excess of $DPNH_2$ is added to the remaining components of the system (FMN, enzyme, and aldehyde),[28] we found that when $DPNH_2$ was added at a low temperature, with the entire system under 8000 psi, the $^1/_2$ rise time to a steady-state level under pressure

was very nearly the same as the $^1/_2$ rise time to a higher steady state when the pressure was released. This result identifies the $^1/_2$ rise time in both cases with $k_2 + k_3$.

Since the aldehyde greatly increases the intensity of luminescence when reduced FMN is added to the enzyme, independently of $DPNH_2$,[21] it follows that aldehyde increases k_2. The mechanism of this effect possibly involves a peroxide.[28]

Although the simplified scheme which has been discussed quantitatively is quite adequate to deal with the saturated system, the low luminescence as well as the quantitatively different effects of pressure when aldehyde is limiting must be accounted for by a somewhat more complicated mechanism. In particular, the facts (1) that there is no depression of steady-state luminescence in the absence of added aldehyde and (2) that the shape of the spike is affected by aldehyde concentration must be taken into account. Granting the simple assumptions that the aldehyde is involved in a further reaction with reduced FMN (not pressure-sensitive), that in its absence $FMNH_2$ accumulates in saturating amounts and is oxidized by another pathway, and that the product of the $FMNH_2$-aldehyde reaction may be utilized either in a luminescent or in a nonluminescent reaction, it is possible to account completely for the results we have observed.

It is of considerable interest that the results of this study with cell-free extracts are fundamentally similar to those obtained earlier with living cells. The general theory that has been developed,[17] primarily from studies of bacterial luminescence in vivo, involved certain assumptions of the sort always necessitated in relatively simple interpretations of a process that can be studied only amid the complexities of intact cells. The results of the present study are consistent with basic assumptions made in the earlier studies with cells. At the same time, they argue for the fundamental similarity of the luminescent system in vivo and in these extracts, whatever the influence of any constituents unessential to light emission but present in the unpurified extracts might be. The quantitative effects of both pressure and temperature are subject to modification by such factors as pH, inorganic salts, and organic substances (e.g., alcohol) in the chemical environment of enzymes, as discussed in detail elsewhere.[17]

The transitory changes, caused by pressure, between steady-state levels of luminescence are of interest with respect to more or less analogous changes in other processes, such as rate of ciliary beating[29] and rhythmicity of cardiac muscle.[30] These processes might be limited by mechanisms similar to those obtaining in luminescence.

Summary.—The intensity of luminescence in cell-free extracts of *A. fischeri* with the addition of reduced diphosphopyridine nucleotide, flavin mononucleotide, and decaldehyde is reversibly affected by hydrostatic pressure, and the changes due to pressure are influenced by temperature both in the same manner and to very nearly the same extent as in the living bacterial cells: steady-state levels are lowered by pressure at temperatures below the normal optimum, are only slightly affected at the optimum, and are raised by pressure at temperatures above the optimum. The influence of pressure and temperature on the inhibition of luminescence by given concentrations of sulfanilamide and alcohol, respectively, is likewise similar in the extracts and cells, in that the sulfanilamide inhibition is scarcely affected by pressure but decreases with rise in temperature, whereas the alcohol inhibition is decreased by pressure and is increased by a rise in temperature. Diminution in luminescence intensity at temperatures above the normal optimum is reversible

on cooling after brief exposures to the high temperatures, and it is partially reversed by pressure, as it is in cells.

Evidence is presented that the luminescent oxidation of flavin mononucleotide by the extracted system, in the presence of decaldehyde, proceeds with a small volume decrease of activation, whereas the reduction of flavin mononucleotide by dihydrodiphosphopyridine nucleotide in this system proceeds with a large volume increase of activation. In the absence of decaldehyde, the intensity of luminescence is much less, and the effects of pressure on the intensity of luminescence are greatly reduced.

Acknowledgment.—Grateful acknowledgment is made to Professor E. Newton Harvey for his active co-operation and interest in this study; to Mr. J. J. Chang for his loan of the photomultiplier and recording apparatus; to Professor Walter J. Kauzmann, both for valuable assistance in the theoretical interpretations and for critical reading of the manuscript; and to Miss Elizabeth Plough for her patient and capable assistance both during the course of the experiments and in analyzing the data.

* These studies have been carried out in part under Contract No. W-7405-eng-26 for the Atomic Energy Commission and were aided by a contract between the Office of Naval Research, Department of the Navy, and Princeton University, NR 165-253, and by funds of the Eugene Higgins Trust allocated to Princeton University.

† Present address: Institute of Radiobiology and Biophysics, University of Chicago, Chicago 37, Illinois.

[1] F. H. Johnson, D. E. Brown, and D. A. Marsland, *Science*, **95**, 200–203, 1942.

[2] D. E. Brown, F. H. Johnson, and D. A. Marsland, *J. Cellular Comp. Physiol.*, **20**, 151–168, 1942.

[3] F. H. Johnson, D. E. S. Brown, and D. A. Marsland, *J. Cellular Comp. Physiol.*, **20**, 269–276, 1942.

[4] F. H. Johnson, H. Eyring, and R. W. Williams, *J. Cellular Comp. Physiol.*, **20**, 247–268, 1942.

[5] H. Eyring and J. L. Magee, *J. Cellular Comp. Physiol.*, **20**, 169–177, 1942.

[6] W. D. McElroy, *J. Cellular Comp. Physiol.*, **21**, 95–116, 1943; F. H. Johnson, E. A. Flagler, R. Simpson, and K. McGeer, *J. Cellular Comp. Physiol.*, **37**, 1–14, 1951.

[7] F. H. Johnson, H. Eyring, and W. Kearns, *Arch. Biochem.*, **3**, 1–31, 1943.

[8] F. H. Johnson, H. Eyring, R. Steblay, H. Chaplin, C. Huber, and G. Gherardi, *J. Gen. Physiol.*, **28**, 463–537, 1945.

[9] F. H. Johnson and D. H. Campbell, *J. Cellular Comp. Physiol.*, **26**, 43–46, 1945; *J. Biol. Chem.*, **163**, 689–698, 1946; F. H. Johnson, M. B. Baylor, and D. Fraser, *Arch. Biochem.*, **19**, 237–245, 1948; D. Fraser, F. H. Johnson, and R. S. Baker, *Arch. Biochem.*, **24**, 314–320, 1949.

[10] A. M. Chase, E. H., Reppert, Jr., and R. M. Ruch, *J. Cellular Comp. Physiol.*, **23**, 27–30, 1944; A. M. Chase, *J. Cellular Comp. Physiol.*, **27**, 121–124, 1946; H. Eyring, F. H. Johnson, and R. L. Gensler, *J. Phys. Chem.*, **50**, 453–464, 1946; F. H. Johnson and I. Lewin, *J. Cellular Comp. Physiol.*, **28**, 1–22, 1946; F. H. Johnson, W. J. Kauzmann, and R. L. Gensler, *Arch. Biochem.*, **19**, 229–236, 1948; E. D. Holyoke and F. H. Johnson, *Arch. Biochem. and Biophys.*, **31**, 41–48, 1951; D. Fraser and F. H. Johnson, *J. Biol. Chem.*, **190**, 417–421, 1951.

[11] H. Koffler, F. H. Johnson, and P. W. Wilson, *J. Am. Chem. Soc.*, **69**, 1113–1117, 1947.

[12] F. H. Johnson and I. Lewin, *J. Cellular Comp. Physiol.*, **28**, 47–75, 77–97, 1946.

[13] C. E. ZoBell and F. H. Johnson, *J. Bacteriol.*, **57**, 179–189, 1949; D. A. Marsland, *J. Cellular Comp. Physiol.*, **36**, 205–227, 1950.

[14] D. H. Campbell and F. H. Johnson, *J. Am. Chem. Soc.*, **68**, 725, 1946.

[15] F. H. Johnson and I. Lewin, *J. Cellular Comp. Physiol.*, **28**, 23–45, 1946; F. H. Johnson and C. E. ZoBell, *J. Bacteriol.*, **57**, 353–358, 359–362, 1949; R. A. C. Foster, F. H. Johnson, and V. K. Miller, *J. Gen. Physiol.*, **33**, 1–16, 1949.

[18] F. H. Johnson and E. A. Flagler, *J. Cellular Comp. Physiol.*, **37**, 15–28, 1951.

[17] F. H. Johnson, H. Eyring, and M. J. Polissar, *The Kinetic Basis of Molecular Biology* (New York: John Wiley & Sons, 1954) (in press).

[18] B. L. Strehler, *J. Am. Chem. Soc.*, **75**, 1264, 1953.

[19] B. L. Strehler and M. J. Cormier, *Arch. Biochem. and Biophys.*, **47**, 16–33, 1953.

[20] M. J. Cormier and B. L. Strehler, *J. Am. Chem. Soc.*, **75**, 4864–4865, 1953; W. D. McElroy, J. W. Hastings, V. Sonnenfeld, and J. Coulombre, *Science*, **118**, 385–386, 1953.

[21] B. L. Strehler, E. N. Harvey, J. J. Chang, and M. J. Cormier, these PROCEEDINGS, **40**, 10–12, 1954.

[22] J. R. Bronk, E. N. Harvey, and F. H. Johnson, *J. Cellular Comp. Physiol.*, **40**, 347–365, 1952.

[23] Diagrammed in F. H. Johnson and F. McK. Schlegel, *J. Cellular Comp. Physiol.*, **31**, 421–425, 1948.

[24] The strain of *A. fischeri* used in many previous studies in Princeton, and the one used in making the extracts, has been lost. Dr. W. D. McElroy kindly supplied the strain used in the present study.

[25] H. Eyring, *J. Chem. Phys.*, **3**, 107–115, 1935; S. Glasstone, K. J. Laidler, and H. Eyring, *The Theory of Rate Processes* (New York: McGraw-Hill Book Co., 1941).

[26] These values were kindly computed by Professor W. J. Kauzmann.

[27] In the present study, high concentrations of sulfanilamide were observed to influence the effects of pressure on the luminescence of cell suspensions of *A. fischeri;* further study of these newly observed effects is contemplated.

[28] B. L. Strehler and M. J. Cormier, *Arch. Biochem. and Biophys.*, 1954 (in press).

[29] D. C. Pease and J. A. Kitching, *J. Cellular Comp. Physiol.*, **14**, 135–142, 1939.

[30] J. Landau and D. A. Marsland, *J. Cellular Comp. Physiol.*, **40**, 367–381, 1952.

REPRESENTATIONS OF THE ANTICOMMUTATION RELATIONS

BY L. GÅRDING AND A. WIGHTMAN

UNIVERSITY OF LUND AND INSTITUTE FOR ADVANCED STUDY; PRINCETON UNIVERSITY

Communicated by E. P. Wigner, May 20, 1954

Introduction.—By a representation of the anticommutation relations we mean a set $\{a_k\}_1^m$ of bounded operators on a separable Hilbert space H, which satisfy

$$a_j a_k + a_k a_j = 0, \qquad a_j a_k^* + a_k^* a_j = \delta_{jk}, \tag{1}$$

when $j, k = 1, \ldots, m$. Jordan and Wigner[1] have proved that if $m < \infty$, every representation is a direct sum of irreducible representations of dimension 2^m which are unitary equivalent. When $m = \infty$, the situation is much more complicated, and von Neumann[2] and Friedrichs[3] have given a variety of essentially inequivalent representations. We shall obtain all representations and give some results on unitary equivalence and irreducibility which show the existence of a maze of inequivalent irreducible representations. The one used in quantum physics, the canonical representation, is characterized by the property that it is the only one for which the number of particles operator $\Sigma\, a_k^* a_k$ exists.

1. *Direct Integrals of Hilbert Spaces.*—Let a set S and a completely additive nonnegative measure μ on S be given. Suppose that ν is a measurable function on S taking the values 1, 2, . . . , and ∞. Consider the set of all (component-wise) measurable functions F on S taking values $F(x) = (F_1(x), F_2(x), \ldots)$ in a Hilbert sequence-space $H_x = H_{\nu(x)}$ of dimension $\nu(x)$, for which

$$|F|^2 = (F, F) = \int_S (F(x), F(x))\, d\mu(x) < \infty.$$

The equivalence classes of these functions modulo those of vanishing norm then constitute what is called the direct integral, $H = \int H_x\, d\mu(x)$, of the Hilbert spaces, H_x, with respect to the measure μ (von Neumann[4]). If $\nu = 1$ a.e., this direct integral coincides with the Hilbert space of all (equivalence classes of) square integrable functions on S. Writing S as a sum of sets S_k on which ν is constant $= k$, we can write H as a direct sum of spaces each of which is a direct integral with constant dimension function ν.

2. *Representations.*—Assume that we have a representation of the anticommutation relations. By virtue of (1), the operators $N_k = a_k^* a_k$ form a commuting set of projections. We first put H into a form which, in a sense, will diagonalize these projections. In fact, let Γ be the set of all infinite sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$, where $\alpha_j = 0$ or 1. Let us put $\Gamma_k = (\alpha;\ \alpha_k = 0)$. By a Borel set in Γ we mean a set constructed from all the sets Γ_k by the usual countable processes. In the sequel, when talking about a measure on Γ, we mean the completion of a totally additive, bounded, not negative, and not identically vanishing set function defined on the Borel sets. Then we can write H as a direct integral, $\int_\Gamma H_\alpha\, d\mu(\alpha)$, over Γ in such a fashion that if f in H corresponds to the function $f(\alpha)$, then $N_k f$ corresponds to the function $\alpha_k f(\alpha)$.

Under component-wise addition mod 2, Γ is a group. The subgroup Δ of Γ which consists of all rational α, i.e., those having at most a finite number of components different from zero, is generated by the elements $\delta_k = (0, \ldots, 0, 1, 0, \ldots)$ with 1 in the kth place. It turns out that the measure μ is a quasi-invariant under Δ, i.e., that $\mu(\alpha)$ and the translated measure $\mu(\alpha + \delta)$, $\delta \in \Delta$, are absolutely continuous with respect to each other and that the dimension function ν is actually invariant under Δ, $\nu(\alpha + \delta) = \nu(\alpha)$, a.e., when $\delta \in \Delta$. Furthermore, the operators

$$A_k = a_k + a_k^*, \qquad B_k = i^{-1}(a_k - a_k^*), \tag{2}$$

are given by the formulas

$$A_k f(\alpha) = (A_k f)(\alpha) = j_k(\alpha)\, c_k(\alpha) \sqrt{\frac{d\mu(\alpha + \delta_k)}{d\mu(\alpha)}}\, f(\alpha + \delta_k),$$

$$B_k f(\alpha) = i^{-1} j_{k+1}(\alpha)\, c_k(\alpha) \sqrt{\frac{d\mu(\alpha + \delta_k)}{d\mu(\alpha)}}\, f(\alpha + \delta_k), \tag{3}$$

where $j_k(\alpha) = (-1)^{\alpha_1 + \cdots \alpha_{k-1}}$ and $\{c_k(\alpha)\}_1^\infty$ are measurable unitary transformations from H_α to $H_{\alpha+\delta_k} = H_\alpha$, satisfying the relations

$$c_k(\alpha + \delta_k) = c_k^*(\alpha),$$

$$c_k(\alpha)\, c_l(\alpha + \delta_k) = c_l(\alpha) c_k(\alpha + \delta e), \tag{4}$$

for almost all α and all k and l.

Conversely, if μ is a quasi-invariant measure on Γ, ν is a measurable invariant dimension function, $H = \int_\Gamma H_\alpha\, d\mu(\alpha)$ is a direct integral of Hilbert sequence spaces H_α of dimension $\nu(\alpha)$, and $\{c_k(\alpha)\}_1^\infty$ is a set of measurable unitary operators from H_α to H_α satisfying (4), then the operators a_k as defined by (2) and (3) are a repre-

sentation of the anticommutation relations. Before turning to the question of the existence of quasi-invariant measures, let us analyze (4).

Let us put $\Gamma_k' = \Gamma_1 \cap \ldots \cap \Gamma_k$. Let $T_k\alpha = (0, \ldots, 0, \alpha_{k+1}, \alpha_{k+2}, \ldots)$ be the projection of α upon Γ_k', and put

$$\gamma_k(\alpha) = c_k(T_k\alpha).$$

Then it follows by a simple calculation from (4) that

$$c_k(\alpha) = \gamma_1{}^{-\alpha_1}(\alpha) \ldots \gamma_{k-1}{}^{-\alpha_{k-1}}(\alpha)\ \gamma_k{}^{(-1)^{\alpha_k}}(\alpha)\ \gamma_{k-1}{}^{\alpha_{k-1}}(\alpha + \delta_k) \ldots \gamma_1{}^{\alpha_1}(\alpha + \delta_k). \tag{5}$$

Conversely, if $\{\gamma_k(\alpha)\}_1^\infty$ are arbitrary unitary measurable transformations from H_a to H_α which are invariant under $\delta_1, \ldots, \delta_k$, then $c_k(\alpha)$ as defined by (5) will satisfy (4). Observing that, if $\{c_k(\alpha)\}_1^\infty$ is one solution of (4), then $\{c_k(\alpha + \gamma)\}_1^\infty$ (γ fixed) is another, we see that $c_k(\alpha)$ is also uniquely determined by its values on $\gamma + \Gamma_k'$, which can be given arbitrarily. Because a representation $\{a_k\}_1^\infty$ is determined by the measure μ, the dimension function ν, and a set of operators satisfying (4), we shall sometimes denote it by $\{\mu, \nu, \{c_k(\alpha)\}_1^\infty\}$.

3. *Quasi-invariant Measures.*—When $m < \infty$, the analogue of Γ is the set of all sequences $(\alpha_1, \ldots, \alpha_m)$ with components $\alpha_k = 0$ or 1. It is clear that a measure μ on this set is always discrete, that it is always quasi-invariant if and only if $\mu(\alpha) > 0$ for all α, and that all quasi-invariant measures are equivalent, i.e., absolutely continuous with respect to each other. When $m = \infty$, this simple situation no longer holds, and there are many inequivalent quasi-invariant measures. A discrete measure on Γ is quasi-invariant if and only if $\mu(\alpha) > 0$ implies $\mu(\beta) > 0$ for all β in the coset $\alpha + \Delta$ of Δ. It is a sum of discrete measures concentrated on different cosets of Δ. Any two discrete quasi-invariant measures concentrated on the same coset are equivalent.

The set Γ is a direct product of a countable number of copies of a set with two points, and we may define a product measure on Γ by putting $\mu(\Gamma) = 1$, $\mu(\Gamma_k) = p_k$, where $0 \leq p_k \leq 1$. Such a measure is quasi-invariant if and only if $0 < p_k < 1$ for all k. It is discrete if and only if $\Pi\, p_k{}^{\alpha_k}(1 - p_k)^{1-\alpha_k} > 0$ for some α, and then it is concentrated on $\alpha + \Delta$. Taking $p_k = {}^1/_2$ for all k, we obtain an invariant measure, Haar measure, on Γ. (Because $\alpha \to \sum_1^\infty \alpha_k 2^{-k}$ maps Γ upon the unit interval in a one-to-one fashion with a countable number of exceptions, any non-discrete measure on Γ determines a measure on the unit interval and conversely. In this mapping, Haar measure on Γ corresponds to Lebesgue measure on the interval.) In order that two product measures defined by $\mu(\Gamma) = 1$, $\mu(\Gamma_k) = p_k$, and $\mu'(\Gamma) = 1$, $\mu'(\Gamma_k) = p_k'$, respectively, be equivalent, it is necessary that $p_k - p_k' \to 0$ as $k \to \infty$ and sufficient that $\sum |p_k - p_k'|\,(1/p_k + 1/(1 - p_k)) < \infty$. This means that we are able to construct a large class of inequivalent quasi-invariant measures.

A quasi-invariant product measure μ also has another important property. It, is ergodic in the sense that if f is bounded and μ-measurable and $f(\alpha + \delta) = f(\alpha)$, a.e., when $\delta \in \Delta$, then f is a constant.

4. *Some Simple Examples.*—Putting $\nu = 1$ and $c_k = 1$ for all k, we obtain the simplest solution of (4). If μ is concentrated on the rationals, we obtain the canonical representation mostly used in quantum physics. We get others, which we will call translated canonical representations, by letting μ be concentrated on a

coset of Δ. Because equivalent measures give unitary equivalent representations and conversely (see the next section) these representations are determined by the cosets up to unitary equivalence.

Other simple solutions of (4) are obtained by putting $\gamma_k = \theta_k$ ($|\theta_k| = 1$), so that $c_k(\alpha) = \theta_k{}^{(-1)^{\alpha_k}}$. More generally, if ν is a constant, we can make γ_k constant and equal to any unitary operator on a sequence space of dimension ν.

5. *Unitary Equivalence.*—Let two representations of the anticommutation relations

$$\{a_k\}_1^\infty = \{\mu, \nu, \{c_k(\alpha)\}_1^\infty\} \text{ and } \{a_k'\}_1^\infty = \{\mu', \nu', \{c_k'(\alpha)\}_1^\infty\}$$

be given. They are said to be unitary equivalent if there exists a unitary operator u from H to H' such that

$$a_k' = ua_ku^*. \tag{6}$$

If, for example, $\{a_k\}_1^\infty$ and $\{a_k'\}_1^\infty$ differ only in the measures μ and μ', and μ is equivalent to μ', then putting

$$Sf(\alpha) = \sqrt{\frac{d\mu(\alpha)}{d\mu'(\alpha)}}\, f(\alpha),$$

we find that S is unitary and that $\{Sa_kS^*\}_1^\infty = \{\mu', \nu, \{c_k(\alpha)\}_1^\infty\}$, so that the two representations are unitary equivalent.

Consider now any two representations which are unitary equivalent. Then (6) implies that

$$N_k' = uN_ku^*$$

for all k. A necessary and sufficient condition for this is that μ be equivalent to μ' and that $\nu = \nu'$ almost everywhere. If this condition is satisfied, we can use the mapping S to map H upon H'. Hence, considering

$$\{s^*a_k's\}_1^\infty = \{\mu, \nu, \{c_k'(\alpha)\}_1^\infty\} \text{ instead of } \{\mu', \nu', \{c_k'(\alpha)\}_1^\infty\},$$

we may suppose from the beginning without loss of generality that $\mu = \mu'$ and $\nu = \nu'$, so that $N_k = N_k'$. Then u is given by the formula

$$uf(\alpha) = u(\alpha)f(\alpha)$$

with a unitary and measurable $u(\alpha)$. Formula (6) is then equivalent to

$$c_k'(\alpha) = u(\alpha)\, c_k(\alpha)\, u^*(\alpha + \delta_k) \tag{7}$$

for almost all α and all k.

First we observe the basic fact that we can satisfy (7) for $k \leq l$ by putting

$$u(\alpha) = u_l(\alpha) = \gamma_1'{}^{-\alpha_1}(\alpha) \ldots \gamma_l'{}^{-\alpha_l}(\alpha)\, \gamma_l{}^{\alpha_l}(\alpha) \ldots \gamma_1{}^{\alpha_1}(\alpha).$$

For a rational α, $u_l(\alpha)$ converges trivially to a unitary transformation $u(\alpha)$, and hence, if μ is concentrated on the rationals, any two representations with the same ν (which is constant in this case) are unitary equivalent. Choosing in particular $c_k' = 1$, we see that the representation in this case is a direct sum of canonical representations. More generally, it can be proved that any representation with a

discrete μ is unitary equivalent to a direct sum of translated canonical representations.

For the nondiscrete measures, a new phenomenon occurs. Here the unitary equivalence class not only depends on μ and ν but may also depend on the terminal behavior of the operators c_k. If, e.g., $\nu = 1$, μ is Haar measure, and $c_k(\alpha) = \theta_k^{(-1)^{\alpha_k}}$, ($|\theta_k| = 1$), and $c_k'(\alpha) = \theta_k'^{(-1)^{\alpha_k}}$, ($|\theta'| = 1$), then the two representations are unitary equivalent if and only if $\Sigma\,|\theta_k - \theta_k'|^2 < \infty$. More generally, let ν be a constant $< \infty$. Let μ have the property that $d\mu(\alpha + \delta_k)/d\mu(\alpha)$ is bounded for almost all α and k (any product measure for which $\mu(\Gamma_k)$ is bounded away from 0 and 1 satisfies this condition), and let $\gamma_k(\alpha)$ and $\gamma_k'(\alpha)$ be constant unitary operators. Put

$$[\gamma_k, \gamma_k'] = \inf |\sigma\gamma_k\sigma^* - \tau\gamma_k'\tau^*|,$$

where σ and τ are unitary and the norm is the ordinary operator norm. Then $[\gamma_k, \gamma_k'] \to 0$, $(k \to \infty)$, is necessary and $\Sigma\,|\sigma\gamma_k\sigma^* - \gamma_k'|^2 < \infty$, ($\sigma$ fixed unitary), is sufficient for unitary equivalence.

6. *Irreducible Representations.*—A representation is said to be irreducible if any bounded linear operator t commuting with all a_k and a_k^* is a multiple of the identity. Any such t can be written in the form

$$tf(\alpha) = t(\alpha)f(\alpha),$$

where $t(\alpha)$ is a uniformly bounded measurable linear operator from H_α to H_α. That t commutes with all a_k and a_k^* is then reflected in the relation

$$t(\alpha)c_k(\alpha) = c_k(\alpha)t(\alpha + \delta_k) \tag{8}$$

for almost all α and all k. Any bounded, measurable, complex-valued function which is invariant under Δ satisfies this relation, and we conclude that in order that a representation be irreducible it is necessary that μ be ergodic. Moreover, because ν^{-1} is invariant, it is also necessary that ν be a constant.

If $\nu = 1$ and μ is ergodic, then it follows from (8) that $t(\alpha)$ is invariant and hence a constant, so that in this case the representation is always irreducible. For an ergodic μ with the property that $d\mu(\alpha + \delta_k)/d\mu(\alpha)$ is uniformly bounded for almost all α and all k, we can construct irreducible representations with a constant finite ν, e.g., by choosing a sequence of constant $\gamma_k(\alpha) = \gamma_k$ such that for an infinite sequence $n_k \to \infty$, $n_{k+1} > n_k + l$, the matrices $\gamma_{nk}, \gamma_{nk+1}, \ldots, \gamma_{nk+l}$ generate all $\nu \times \nu$ matrices. This, combined with our results on unitary equivalence, proves that for any $\nu < \infty$ there exist infinitely many irreducible inequivalent representations. It is not known whether irreducible representations with $\nu = \infty$ exist.

[1] P. Jordan and E. Wigner, *Z. Phys.*, **47**, 631, 1928.

[2] J. von Neumann, *Composit. Math.*, **6**, No. 1, 75–76, 1938.

[3] K. Friedrichs, *Mathematical Aspects of the Quantum Theory of Fields* (New York: Interscience Publishers, Inc., 1953).

[4] J. von Neumann, *Ann. Math.*, **50**, 401, 1949.

REPRESENTATIONS OF THE COMMUTATION RELATIONS

By L. Gårding and A. Wightman

UNIVERSITY OF LUND AND INSTITUTE FOR ADVANCED STUDY; PRINCETON UNIVERSITY

Communicated by E. P. Wigner, May 20, 1954

Introduction.—In a previous note we have given some results on representations of the anticommutation relations of quantum physics.[1] The object of the present note is to give similar results for the commutation relations. For a definition of direct integrals of Hilbert spaces which will be used below we refer the reader to our previous note and the paper by J. von Neumann.[2]

Roughly speaking, a representation of the commutation rules is two sets of unbounded self-adjoint operators, $\{p_k\}_1^m$ and $\{q_k\}_1^m$ on a separable Hilbert space H satisfying

$$\begin{aligned} p_k q_j - q_j p_k &= i^{-1}\delta_{jk}, \\ q_k q_j - q_j q_k &= 0, \\ p_k p_j - p_j p_k &= 0. \end{aligned} \tag{1}$$

In order to avoid difficulties with the domains of these operators, we introduce with H. Weyl[3] the operators $U(\alpha) = \exp i \Sigma \alpha_k p_k$ and $V(\alpha) = \exp i \Sigma \alpha_k q_k$, where the numbers α_k are real and all but a finite number of them vanish if $m = \infty$. Then $\{U(\alpha)\}$ and $\{V(\alpha)\}$ are two sets of unitary operators on H which are continuous functions of the components α_k of α and satisfy

$$\begin{gathered} U(0) = V(0) = 1, \\ U(\alpha)U(\alpha') = U(\alpha + \alpha'), \qquad V(\alpha)V(\alpha') = V(\alpha + \alpha'), \\ U(\alpha)V(\beta) = \exp i\, \alpha\cdot\beta V(\beta)U(\alpha) \qquad (\alpha\cdot\beta = \textstyle\sum \alpha_k\beta_k). \end{gathered} \tag{2}$$

Two sets of unitary operators $\{U(\alpha)\}$ and $\{V(\alpha)\}$ with these properties will be called a representation of the commutation rules. The infinitesimal generators $p_k = (1/i)\ (\partial U(\alpha)/\partial\alpha_k)_{\alpha=0}$ and $q_k = (1/i)(\partial V(\alpha)/\partial\alpha_k)_{\alpha=0}$ are then densely defined and self-adjoint, and (1) is true on a dense subset of H.

The problem of finding all representations was solved by von Neumann[4] when $m < \infty$. He found that every representation is a direct sum of irreducible representations all equivalent to each other. For $m = \infty$, Friedrichs[5] has found some essentially inequivalent representations. We shall consider this case more closely, give all representations, and state some results on irreducibility and unitary equivalence which show that here, as in the case of the anticommutation relations, there exists a maze of irreducible inequivalent representations.

1. *The Representations.*—On the common domain of definition of p_k and q_k put

$$a_k = \frac{1}{\sqrt{2}}\,(p_k - iq_k)$$

and

$$a_k^* = \frac{1}{\sqrt{2}}\,(p_k + iq_k).$$

By virtue of von Neumann's results, a_k^* is then the actual adjoint of a_k, and $a_k^{**} = a_k$. Moreover, $a_k a_k^*$ is defined on a dense subset of H and has the form

$$a_k a_k^* = \sum_{l=0}^{\infty} (l+1) P_k^l,$$

where the P_k^l are commuting projections which satisfy

$$P_k^j P_k^l = \delta_{jl} P_k^l$$

and

$$\sum_{l=0}^{\infty} P_k^l = 1.$$

Let I be the set of all infinite sequences $n = (n_1, n_2, \dots)$ with integral components $n_k \geq 0$. It is a semigroup under component-wise addition. Let Δ denote the set of "rational" sequences, i.e., those with only a finite number of components different from zero. It is generated by the sequences $\delta_k = (0, \dots, 0, 1, 0, \dots)$, with 1 in the kth place. Let I_{kj} be the set of all n in I for which $n_k = j$. By a Borel set in I we mean a set constructed from all the sets I_{kj} by the usual countable processes. In the following, when speaking of a measure on I, we mean the completion of a totally additive, bounded, not negative, and not identically vanishing set function defined on the Borel sets.

For a given representation we can always write H as a direct integral over I, $\int_I H_n d\mu(n)$, which diagonalizes the projections P_k^j in such a fashion that if f corresponds to the function $f(n)$, then P_k^j corresponds to the function $\delta_{jnk} f(n)$. The measure μ is quasi-invariant in the sense that $\mu(n)$ and the translated measure $\mu(n+\delta)$ are absolutely continuous with respect to each other for all δ in Δ. The dimension $\nu(n)$ of the Hilbert space H_n is invariant under Δ, $\nu(n+\delta) = \nu(n)$ for almost all n and all δ in Δ. The operators a_k and a_k^* are given by

$$\begin{aligned}(a_k f)(n) = a_k f(n) &= \sqrt{n_k + 1}\, c_k(n) f(n + \delta_k) \sqrt{\frac{d\mu(n+\delta_k)}{d\mu(n)}} \\ a_k^* f(n) &= \sqrt{n_k}\, c_k^*(n - \delta_k) f(n - \delta_k) \sqrt{\frac{d\mu(n-\delta_k)}{d\mu(n)}} \end{aligned} \tag{3}$$

Here $c_k(n)$ are measurable unitary operators from H_n to $H_{n+\delta_k} = H_n$, satisfying

$$c_k(n) c_l(n+\delta_k) = c_l(n) c_k(n+\delta_l). \tag{4}$$

The domain of definition D_k of a_k and a_k^* consists of all f for which

$$\int n_k |f(n)|^2 \, d\mu(n) < \infty.$$

Conversely, if $H = \int_I H_n d\mu(n)$ is a direct integral with a quasi-invariant measure μ and an invariant dimension function ν, if $\{c_k(n)\}_1^\infty$ are measurable unitary operators satisfying (4), and if a_k and a_k^* are defined on D_k by (3), then the operators p_k and q_k defined on D_k by

$$p_k = \frac{1}{\sqrt{2}} (a_k + a_k^*)$$

and

$$q_k = \frac{i}{\sqrt{2}} (a_k - a_k^*)$$

have self-adjoint closures and generate sets $\{U(\alpha)\}$ and $\{V(\alpha)\}$ of unitary operators which constitute a representation of the commutation relations.

Because a representation is determined by the measure μ, the dimension function ν, and a set of operators satisfying (4), we shall sometimes denote it by $\{U(\alpha), V(\alpha)\} = \{\mu, \nu, \{c_k(n)\}_1^\infty\}$.

Postponing for the moment the question of the existence of quasi-invariant measures, let us analyze (4). Put $I_{k0} = (n;\ n_k = 0)$ and let $T_k n = (0, \ldots, 0, n_k, n_{k+1}, \ldots)$ be the projection of n upon $I'_{k-1} = I_{10} \cap \ldots \cap I_{k-1,0}$. Then it follows from (4) that we can express $c_k(n)$ in terms of $\gamma_k(n) = c_k(T_k n)$. In fact, a straightforward calculation shows that

$$c_k(n) = \pi_1(n) \ldots \pi_{k-1}(n) c_k(T_k n) \pi_{k-1}{}^*(n + \delta_k) \ldots \pi_1{}^*(n + \delta_k), \tag{5}$$

where

$$\pi_k(n) = 1 \qquad (n_k = 0),$$

$$\pi_k(n) = c_k{}^*(T_k n - \delta_k) \ldots \qquad c_k{}^*(T_k(n - n_k\delta_k)) \qquad (n_k > 0) \tag{6}$$

Conversely, if $c_k(T_k n) = \gamma_k(n)$ are arbitrary unitary matrices which are invariant under $\delta_1, \ldots, \delta_{k-1}$ then $c_k(n)$ as defined by (5) and (6) will satisfy (4).

There are similar formulas expressing $c_k(n)$ in terms of $\gamma_k{}^{(r)}(n) = c_k(r_1, \ldots, r_{k-1}, n_k, \ldots)$ for an arbitrary but fixed r in I.

2. *Quasi-invariant Measures.*—When $m < \infty$, the analogue of I is the set I_m of all sequences $(n_1, \ldots, n_m)$ with integral components $n_k > 0$. It is clear that a measure μ on I_m is always discrete, that it is quasi-invariant if and only if $\mu(n) > 0$ for all n, and that all quasi-invariant measures are equivalent, i.e., absolutely continuous with respect to each other. When $m = \infty$, this simple situation no longer holds, and there are many inequivalent quasi-invariant measures. For a given r in I let Δ_r denote its "coset" with respect to Δ, i.e., the set of all n in I for which $n_k = r_k$ for all but a finite number of k's. A discrete measure on I is quasi-invariant if and only if $\mu(r) > 0$ implies that $\mu(n) > 0$ for all n in Δ_r. In a natural way it splits up into a countable number of discrete measures, each concentrated on some Δ_r. Any two such measures are equivalent.

As a set, I is a direct product of copies of the natural numbers. Hence, putting $I_{kj} = (n;\ n_k = j)$, we may define a product measure on I, putting $\mu(I_{kj}) = p_{kj} \geq 0$, where $\mu(I) = \Sigma_{j=0}^\infty p_{kj} = 1$. Such a measure is quasi-invariant if and only if all $p_{kj} > 0$. It is discrete if and only if, for some m, $\Pi p_{kr_k} > 0$, and then it is concentrated on Δ_r. The question of equivalence of quasi-invariant product measures is more complicated than in the case of the anticommutation relations. A rather special result is the following: Two measures, μ and μ', for which $p_{kj} = p_j$ and $p_{kj}{}' = p_j{}'$ are independent of k, are inequivalent if the set $\{p_j{}^{-1}p_j{}'\}_1^\infty$ does not have 1 as a limit point. At any rate, this result shows the existence of a large class of inequivalent quasi-invariant measures.

All quasi-invariant product measures μ are ergodic in the sense that if f is a bounded, invariant μ-measurable function, it is a constant.

3. *Some simple Examples*—The simplest solution of (4) is obtained by putting $c_k(n) = 1$ for all k. Putting $\nu = 1$ and letting μ be concentrated on the rationals, we obtain essentially the canonical representation mostly used in quantum physics.

A different choice of μ, as long as it is concentrated on the rationals, does not change the unitary equivalence class of the representation, because equivalent measures give unitary equivalent representations (see the next section). The canonical representation is the only one for which the number of particles operator $\Sigma\, a_k^* a_k$ exists. Letting μ be concentrated on a Δ_r, we obtain a translated canonical representation. We can also satisfy (4) by putting $c_k(n) = \theta_k$, with $|\theta_k| = 1$, and combine this choice with any ν and μ.

4. *Unitary Equivalence.*—Let two representations of the commutation relations $R = \{U(\alpha), V(\alpha)\} = \{\mu, \nu, \{c_k(n)\}_1^\infty\}$ and $R' = \{U'(\alpha), V'(\alpha)\} = \{\mu', \nu', \{c_k'(n)\}_1^\infty\}$ be given. They are said to be unitary equivalent if a unitary operator W from H to H' exists so that

$$U'(\alpha) = WU(\alpha)W^*, \qquad V'(\alpha) = WV(\alpha)W^*. \tag{7}$$

If, for example, R and R' differ only with respect to the measures μ and μ', and μ is equivalent to μ', then, putting

$$Sf(n) = \sqrt{\frac{d\mu(n)}{d\mu'(n)}}\, f(n),$$

we find that S is unitary and that $\{SU(\alpha)S^*, SV(\alpha)S^*\} = \{\mu', \nu, \{c_k(n)\}_1^\infty\} = R'$. so that the two representations are unitary equivalent.

Consider any two representations R and R' which are unitary equivalent. Then (7) implies that

$$P'_k{}^j = WP_k{}^jW^*,$$

and for this it is necessary and sufficient that μ and μ' are equivalent and that $\nu = \nu'$ a.e. If this condition is satisfied, we can use the mapping S to map H upon H', and, considering

$$\{S^*U'(\alpha)S, S^*V'(\alpha)S\} = \{\mu, \nu, \{c_k'(n)\}_1^\infty\}$$

instead of R', we may suppose from the beginning, without loss of generality, that $\mu = \mu'$, $\nu = \nu'$, so that $P_k{}^j = P'_k{}^j$. In this case, W is given by

$$Wf(n) = w(n)f(n), \tag{8}$$

where $w(n)$ is a measurable unitary operator from H_n to H_n, satisfying

$$c_k'(n) = w(n)c_k(n)w^*(n + \delta_k) \tag{9}$$

for almost all n and all k. Conversely, if a set $\{w(n)\}$ of measurable unitary operators satisfies this equation, W as defined by (8) will satisfy (7), so that the two representations are unitary equivalent.

For $k \leq l$ we can satisfy (9) by putting

$$w(n) = w_l(n) = \pi_1'(n) \ldots \pi_{l-1}'(n)\, \pi_{l-1}{}^*(n) \ldots \pi_1{}^*(n),$$

where $\pi_k(n)$ is given by (6) and analogously for $\pi_k'(n)$. If now μ is concentrated on the rationals, the operators $w_l(n)$ converge trivially to a unitary operator as

$l \to \infty$. Hence, choosing in particular $c_k' = 1$, we find that the representation is equivalent to a direct sum of copies of the canonical representation. More generally, it can be shown that any representation with a discrete μ is a direct sum of translated canonical representations.

When μ is not discrete, the unitary equivalence class not only depends on μ and ν but may also depend on the terminal behavior of the operators c_k. Let us, for example, consider the case where $\nu = 1$ and the measure μ is a product measure given by $\mu(I_{kj}) = 2^{-j-1}$. Then, for the two representations given by $c_k(n) = \theta_k$ and $c_k'(n) = \theta_k'$, respectively, to be unitary equivalent, it is necessary that $|\theta_k \to \theta_k'| \to 0$ as $k \to \infty$ and sufficient that $\Sigma|\theta_k - \theta_k'|$ converge. More generally, with the same μ and constant $\nu < \infty$, let two representations be given with constant $c_k(T_k n) = \gamma_k$ and $c_k'(T_k n) = \gamma_k'$, respectively. If $[\gamma_k, \gamma_k'] = \inf |u\gamma_k u^* - v\gamma_k' v^*|$ (u, v unitary, operator norm), then $[\gamma_k, \gamma_k'] \to 0$ as $k \to \infty$ is necessary, and $\Sigma|\gamma_k - u\gamma_k' u^*| < \infty$ for some fixed unitary u is sufficient for unitary equivalence. The same results are true where μ is replaced by a product measure for which $\mu(I_{kj})/\mu(I_{kj+1})$ is bounded away from 0 and ∞ for all k and j.

5. *Irreducible Representations.*—A representation is called irreducible if any bounded operator which commutes with all $U(\alpha)$ and $V(\alpha)$ is a constant. Any such operator has the form

$$Tf(n) = t(n)f(n), \tag{10}$$

where $t(n)$ is a measurable bounded operator from H_n to H_n and satisfies

$$t(n)c_k(n) = c_k(n)t(n + \delta_k) \tag{11}$$

for almost all n and all k. Conversely, if T is defined by (10), where $t(n)$ is measurable and bounded and satisfies (11), T commutes with all $U(\alpha)$ and $V(\alpha)$.

It is clear that any invariant, bounded, complex-valued function satisfies (11), and hence, for a representation to be irreducible, it is necessary that μ be ergodic. Moreover, because $\nu^{-1}(n)$ is invariant and bounded, it is also necessary that ν be a constant. If $\nu = 1$, then $t(n)$ has complex values, and it follows from (11) that the ergodicity of μ is sufficient to guarantee irreducibility.

If μ is a product measure for which $\mu(I_{kj})/\mu(I_{kj+1})$ is bounded, we can always construct irreducible representations with a given $\nu < \infty$ by choosing unitary matrices $h_1 \ldots h_s$ which generate all $\nu \times \nu$ matrices and putting $c_{k+l}(T_{k+l}n) = \gamma_{k+l} = h_l$ ($l = 1, \ldots, s$) when k runs through a sequence $k_1, k_2, \ldots \to \infty$ for which $k_{j+1} - k_j > s$. This, combined with the results of the preceding section, shows the existence of a large number of irreducible inequivalent representations with a suitable given μ and any finite ν. It is not known whether irreducible representations with $\nu = \infty$ exist.

[1] L. Gårding and A. Wightman, these PROCEEDINGS, **40**, 617–621, 1954.

[2] J. von Neumann, *Ann. Math.*, **50**, 401, 1949.

[3] H. Weyl, *Z. Phys.*, **46**, 1, 1927.

[4] J. von Neumann, *Math. Ann.*, **104**, 570, 1931.

[5] K. O. Friedrichs, *Mathematical Aspects of the Quantum Theory of Fields* (New York: Interscience Publishers, Inc., 1953).

NOTE ON A "PERMANENT" EXPERIMENTAL ALTERATION OF GENETIC CONSTITUTION IN A NATURAL POPULATION

By Caryl P. Haskins and Edna F. Haskins

UNION COLLEGE, SCHENECTADY, N.Y.

Communicated by Th. Dobzhansky, March 12, 1954

The definition of balanced polymorphism in an animal population (Ford[1]) implies the coexistence of two or more distinguishable genotypes in a population and the maintenance of an approximately stabilized ratio of their numbers over considerable periods of time. Investigations of such systems are now very numerous. The studies of Bellamy[2] in the grouse locust *Tettigidea* long ago introduced the concept of a system of this kind, and they have been amplified by the later work of Nabours,[3] Nabours and Snyder,[4] Nabours, Larson, and Hartwig,[5] Rubtzov,[6] and Good[7] in the Tettigidae, to mention only a few. De Zulueta[8] described a similar polymorphism in the beetle *Phytodecta variabilis*. Timofeef-Ressovsky[9] studied the polymorphism, involved two monogenic color patterns, in the beetle *Adalia bipunctata*, and Shull[10] has given further consideration to this form. Tan,[11] Komai, Chino, and Hosino[12] and Komai and Hosino[13] have published rather extensive investigations of another polymorphic beetle, *Harmonia axyridis*. Dobzhansky[14] investigated the polymorphism of a lady beetle, *Sospita vigintiguttata*, and his later studies of balanced chromosome polymorphism in *Drosophila* are classic. The work of Gerould[15] and Hovanitz[16] on polymorphism in the butterfly genus *Colias;* of Kinsey[17] in the gall wasp *Biorrhizia eburnea;* of Gordon[18] on the Poeciliid fish *Platypoecilus maculatus;* of Diver,[19, 20] Fisher and Diver,[21] and Cain and Sheppard[22] on the polymorphic snails of the genus *Cepea*—to mention but a very few diverse examples—have demonstrated how widespread is the occurrence of polymorphic systems which, over the long term, may be described as stable.

In a few cases, it has been possible to investigate the stability of the gene ratios in such populations over very considerable periods. Thus Kinsey's observations extended over ten years; Gordon's data were directly comparable to information provided by collections of Meek made thirty and thirty-seven years earlier and by a small collection taken seventy years before the last group; Gerould was able to compare the ratio of white to yellow females of *Colias philodice* taken at a particular location at Hanover, New Hampshire, in 1911 and again in 1940 and found it unchanged. The classic case is that reported by Diver for *Cepea*,[19] in which the ratios of various of the polymorphically colored forms coincided closely with those found in fossil deposits of the species of Pleistocene age.

Such systems of balanced polymorphs in wild populations continue to be actively studied at present, and much new information about their composition and stability is constantly appearing. However, no investigations appear to have been made of another phase of this problem—the reaction of such a natural balanced polymorphic population to the artificial introduction of a novel genetic complex as it becomes incorporated permanently in the system.[23] This, of course, is a "laboratory" representation of the situation which must occur in every natural balanced polymorphic population when a new mutation arises which ultimately becomes fixed. This obvious gap in information seemed sufficiently important to war-

rant some effort to fill it. It is the purpose of this note to give a very brief preliminary account of a study of this sort begun in certain wild populations of the viviparous Poeciliid fish *Lebistes reticulatus*, leaving more detailed description for a later time.

Although the occurrence of new genetic changes and their permanent incorporation as balanced factors represent the elementary processes in the formation of every balanced polymorphic population, several circumstances may make an artificial simulation of this event in wild populations somewhat difficult to accomplish. Prominent among these is the occurrence of short-range instability of gene ratios in the population or simply sharp short-term fluctuations in the size of the population itself. Recurrent short-range fluctuations are very common in populations of short-lived, rapid-breeding species which live in environments subject to cycles of sharp variability and where features of the external environment exert strong selective balancing influences. Dobzhansky,[14] Timofeef-Ressovsky[9], and Tan,[11] Komai, Chino, and Hosino,[12] and Komai and Hosino[13] all reported marked short-term fluctuations of this sort in the beetle genera *Sospita*, *Adalia*, and *Harmonia*, respectively, and the annual fluctuation both of the ratios of chromosome types and of actual population numbers in temperate-living *Drosophila* illustrate this situation particularly vividly. Hovanitz[16] has recently emphasized that the striking long-term constancy in the ratio of the female color forms in populations of *Colias* may mask very large shifts from day to day or even from season to season. In animals with these characteristics, it is likely that a gene artificially introduced into a wild population will be eliminated rather shortly, during a period when its own frequency drops very low or when the population itself becomes drastically reduced in numbers.

Thus short-term as well as long-term stability in a polymorphic population is a reasonable requirement if studies of artificially introduced genes are to be practicable. A second important requirement is that there shall be available in the species one or several genes whose adaptive values lie somewhere near those already present in the population and which, though not present in the experimental population are known to be coadapted to its genotype.

It is not difficult, indeed, to sum up the characteristics of a polymorphic organism which theoretically should make it adaptable for experiments of this kind, though it may not be possible to fulfil them all. The organism should (1) live in an environment with a minimum of seasonal fluctuation; (2) show minimal cyclic fluctuation in the proportion of its known polymorphic genes; (3) show minimal fluctuation in the total size of its populations (not subject to violent annual fluctuations of numbers); (4) be basically nonmigratory and not subject to great migration pressure; (5) exist naturally in populations which (*a*) are effectively isolated and (*b*) lend themselves readily to estimates of magnitude (as by release-recapture); (6) have a long breeding life relative to its period of immaturity and relative to such seasonal environmental fluctuations as are present; (7) breed randomly throughout the year; and (8) offer a reservoir of "marked" genotypes in which (*a*) the "markers" behave as dominants, (*b*) the "markers" are conspicuous, permitting ready identification in natural environments, and (*c*) the "markers" are simply and clearly expressed in inheritance and include genes that, though absent from certain wild populations, are nevertheless coadapted to them.

This rather exacting set of requirements is fulfilled remarkably well by the small vivaparous Poeciliid fish *L. reticulatus*, the common aquarium "guppy." *Lebistes* is native to fresh-water streams and ponds and to brackish marginal lagoons of the Lesser Antilles, Trinidad, and Tobago, Venezuela, and through Brazil as far south as Santos. It has been widely naturalized in waters of other tropical regions of both hemispheres as an agent of mosquito control. In Trinidad, where the present studies are in progress, it is an abundant inhabitant of fresh-water streams, many of which do not vary radically in a cyclic manner in temperature, volume, turbidity, or various other physical or biological characteristics the year around. In such stable locations the populations show little annual cyclic fluctuation in numbers, and such studies as have been made of the frequency of individual genetic elements in specific populations suggest considerable constancy over the short as well as over the longer term.

The species is essentially nonmigratory, and collateral studies give considerable evidence that the average breeding range of an individual when undisturbed is remarkably small. Many naturally occurring populations can be found that are quite effectively isolated from migration pressure from adjacent regions, either by positive physical barriers or often almost, if not quite, as effectively by mere distance. The magnitude of such populations can be readily estimated by a variant of the release-recapture method, released individuals being marked by nicks in the caudal fin.

Lebistes reticulatus exhibits marked sexual dimorphism. Wild females are always essentially without secondary sexual coloration (though by no means genetically empty of factors for such coloring). They are plain gray fish, with darker reticulate patterning, measuring approximately 2 inches long at maturity. Mature males are considerably smaller and are marked with brilliant color patterns. In this patterning they exhibit perhaps the most extreme color polymorphism in nature—so extreme that it is difficult to find two male individuals in a wild population which are patterned alike. These patterns can be resolved into elements which are under simple genetic control, a considerable number of which have been analyzed by Schmidt,[24] Blacher,[25] Goodrich *et al.*,[26] and others, but most extensively by Winge and by Winge and Ditlevsen.[27, 28] Much further work remains to be done in the analysis of the pattern genes of wild populations of *Lebistes*, but it is evident beyond much doubt at present that in Trinidad at least all these genes are sex-limited in character and that all of them behave as dominants in the male, though their degree of penetrance may vary considerably. Several of them are quite conspicuous and very suitable as "markers." The fact that closely neighboring populations may differ decidedly in their content of pattern genes, elements which are absent in one population often being present in quite high frequency in a neighboring one, suggests that in Trinidad all the color genes which are found are coadapted to a general island genetic system, though perhaps none is represented in every population.

Lebistes is an ovoviviparous form, with a gestation period of approximately 1 month and a developmental period from birth to maturity of approximately 3 months. Individuals have a breeding potential of 2–3 years, though this may seldom be realized in nature. Breeding occurs throughout the year, and it is the general rule for overlap broods to develop continuously in a population. Thus *Le-*

bistes fulfils the specified conditions rather better than could reasonably be expected, and it was chosen for the present studies.

After some preliminary exploration and testing, three populations were selected at three stream locations in the Northern Range of Trinidad. The general characteristics of this region have already been described.[29] All the locations were fairly constant in water flow and volume throughout the year, and they were situated within a circle of a radius of 5 miles. In other features, however, they showed sharp contrasts. The first location was on the Arima River, about $3^1/_2$ miles above the town of Arima. The river at this point forms a wide, relatively heavily shaded stream flowing over beds of dark impervious rock and gravelly schists, with an average depth of about 12 inches but with frequent pools 18–24 inches deep. The pH of the water (6.7) and both calcium and magnesium content (Ca 11.2 p.p.m., Mg 39.0 p.p.m.) are low relative to certain more easterly locations. The stream is heavily populated with natural predators of *Lebistes*, both vertebrate (largely fish, including such species as *Aequidens latifrons* and other Cichlids and *Pristella riddleyi* and other carnivorous Characins) and invertebrate (largely Crustacea). In this environment, *Lebistes* forms scattered colonies, frequently of 50–100 individuals, along the banks in specific locations, often separated by many yards of "empty" water. Observations on marked fish indicate that the members of these populations are quite tenacious of their locations and that interchange of individuals is surprisingly deliberate, even between fairly closely neighboring groups. The color patterns of the males are unusually pale and inconspicuous, with red and yellow elements decidedly deficient.

The second population was chosen at a point in the Aripo River, some 5 miles east of that in the Arima. The Arima and the Aripo are parallel streams flowing southward off the Northern Range into a common river, the Caroni. The environment of the Aripo contrasted rather markedly with that of the Arima. The stream at this point flows over limestone outcrop, and pH and calcium and magnesium content are higher (pH 7.2, Ca 15.0 p.p.m., Mg 48.0 p.p.m.). The bottom is light-colored gravel, the water well aerated, very clear, and brilliantly sunlit. Fish predators are limited to a single species (*Rivulus hartii*, also present in the Arima). Predatory Crustacea are present in approximately the same density as in the Arima. The *Lebistes* population is much denser and more continuous, and the males are phenotypically the most highly colored found in Trinidad.

The third location selected was on the Marianito River, a northward-flowing stream taking its headwaters about $2^1/_2$ miles north and slightly west across the watershed from the headwaters of the Arima and emptying through a larger stream into the sea on the northern coast. In many respects conditions here were intermediate between those of the Aripo and the Arima. Stream bottom was of yellow-brown mud and gravel, water flow and volume quite constant (except in the last year of observation) and partially shaded (densely shaded in the last year). Only two species of fish predators were found—*R. hartii* and an as yet unidentified blennid. Crustacean predators were abundant. The population was continuous and dense (except in the last year). The males were rather highly and very characteristically colored, differing from those of both the other populations in the intensity and extent of red pigments.

The genetic complex selected for introduction into these populations was isolated

in the laboratory from Danish aquarium stock by Winge in 1916 and described some what later,[27] being designated by him as *Maculatus*. Stock was obtained from Winge through his kindness in 1934 and has been carried in small mass cultures in our laboratory since that time. It is phenotypically expressed as a complex of two principal color elements, both conspicuous. The first is an intense black patch in the usually colorless dorsal fin, the second, a sharply defined red lateral body spot located almost directly under the dorsal fin insertion. These elements both behave as though located in the *Y* chromosome, as Winge determined[27] and we have amply confirmed. They are thus dominants, normally inherited directly from male to male, irrespective of the female parent. Crossover between the factors occurs with extreme rarity.

In practice, this complex proved exceedingly useful. Not only is it conspicuous and easily identified by inspection when introduced into wild populations with uncolored dorsal fins, but the lateral red spot provides insurance against confusing

TABLE 1

SAMPLES OF MALES CAPTURED AND EXAMINED IN THREE WILD POPULATIONS BEFORE INTRODUCTION OF THE *Maculatus* COMPLEX

DATE TAKEN	NUMBER WITHOUT DORSAL MELANIN	NUMBER WITH DORSAL MELANIN	NUMBER OF *Maculatus*
	Arima River		
March 17–25, 1947	536	0	0
April 3, 1950	15	0	0
April 14, 1951	111	1	0
Total	662	1	0
	Marianito River		
March 17–25, 1947	220	0	0
April 13–15, 1951	272	0	0
Total	492	0	0
	Aripo River		
April 1–2, 1950	358	0	0
December 28, 1950	206	0	0
August 7–23, 1951	341	0	0
December 30, 1952	286	0	0
Total	1,191	0	0

the introduced genotype with similar natural ones which might enter from a neighboring population under migration pressure. For though gene complexes involving closely similar coloration of the dorsal fin occur in relatively high percentages in many Trinidad populations of *Lebistes*, the particular combination of body and fin coloration characteristic of *Maculatus* has never been found, in some seven thousand specimens examined, except in artificially "infected" populations.

As a preliminary measure, all three experimental populations were examined with some care at intervals over several years to insure that patterns similar to or identical with that to be introduced were not present. At each examination they were first rather thoroughly inspected, using simple viewing equipment which eliminated scattered light from the water surface, and depending upon natural illumination. This procedure was proved quite reliable in preliminary tests with populations into which known "contaminations" had been introduced. Following this preliminary inspection, samples of males were captured and examined more closely. In only one case was an individual with melanin in the dorsal fin found

among the 2,345 males so examined from the three populations over a total period of five years, and the *Maculatus* pattern was never found. These preliminary counts are shown in Table 1.

"Marked" stock was prepared for introduction into the Arima and Marianito locations by crossing laboratory-bred males containing *Maculatus* to females of wild stock from the two populations and twice backcrossing the FI male progeny to similar females. The final foundation stock was thus "marked" with the *Maculatus* phenotype (often in combination with additional patterns inherited through the females) but was seven-eighths of the genotype of the population into which it was to be introduced. This stock was then bred to considerable numbers by Walter Straub, of Micco, Florida, and was shipped to Trinidad, where it was introduced into the experimental populations. In the case of the Aripo population, the backcrossing procedure was omitted and laboratory stock introduced directly.

The Arima and Marianito populations were then examined at intervals over a 3-year period to establish whether the introduced gene complex had become incorporated as a "permanent" component in the population genotype and also to follow its course both in dispersion along the stream and in given subpopulations. The Aripo population was similarly examined for the one year that it has been under observation. The methods of examination were essentially similar to those used in the preliminary work, except that all "marked" individuals, as they were located, were captured and examined in a glass vessel to check for the characteristic body spot, in addition to the dorsal marking. Such "marked" males proved much easier to locate than was expected. The numbers so found are recorded in Table 2, A and B. They represent a minimum of the number actually present at the times of examination, since some individuals may well have been missed, though all practicable care was used to make the recoveries complete.

The primary purpose of the experiment was to gain some measure of the rate and final extent of dispersion of individuals and genotypes in essentially two-dimensional stream populations. This work is still in progress and will later be reported. It is already clear, however, that the introduced *Maculatus* complex has persisted in the Arima and Marianito populations for 32 months after its introduction and in the Aripo for 12. This corresponds to approximately 10 generations in the first two cases and to 4 in the third, and may imply the attainment of a new point of balance in the population after its artificial alteration.

In the Arima population there is indication that a genuine equilibrium has, in fact, been reached. In the Marianito there has been a continuous decline in the frequency of the *Maculatus* population over the 32-month period, and especially during the past year. This coincides with a sudden recent change in the character of the location itself, involving much channel scouring, apparently due to flash flooding and progressively heavier shading due to pronounced growth of secondary marginal vegetation—conditions unfavorable to *Lebistes*. These changes have been accompanied by a marked drop in the population density and a generally suboptimal aspect of the population itself. Whether the population, and the equilibrium of the *Maculatus* complex in it, will become re-established remains to be seen.

Our great appreciation is expressed to Professor Th. Dobzhansky, not only for his reading and critical review of the present paper but for many suggestions and

much advice and assistance which have been invaluable throughout the whole course of the work.

TABLE 2

A. SAMPLES OF *Maculatus* MALES SUBSEQUENTLY RECOVERED IN THREE POPULATIONS

DATE OF EXAMINATION	TOTAL NUMBER OF *Maculatus* MALES RECOVERED
	Arima River*
April 17, 1951	140
April 18, 1951	135
April 19, 1951	92
April 20, 1951	98
April 21, 1951	83
August 16–25, 1951	27
December 25–29, 1951	7
December 19–29, 1952	14
December 23–28, 1953	13
	Marianito River†
April 17, 1951	71
April 18, 1951	63
April 19, 1951	53
April 21, 1951	73
August 7–17, 1951	40
December 25–30, 1951	18
December 18–30, 1952	13
December 22–27, 1953	5
	Aripo River‡
December 24, 1953	5

TABLE 2

B. SAMPLES OF *Maculatus* MALES SUBSEQUENTLY RECOVERED AT POINTS OF INTRODUCTION IN THREE POPULATIONS

DATE OF EXAMINATION	NUMBER OF *Maculatus* MALES RECOVERED	ESTIMATED POPULATION AT "ORIGIN"
	Arima River	
April 17, 1951	140	60
April 18, 1951	74	60
April 19, 1951	47	60
April 20, 1951	53	60
April 21, 1951	33	60
August 17, 1951	2	60
December 25, 1951	2	60
December 19, 1952	2	68 (Counted)§
December 22, 1953	4	73 (Counted)§
	Marianito River	
April 17, 1951	38	200
April 18, 1951	31	200
April 19, 1951	31	200
April 21, 1951	29	200
August 7, 1951	5‖	200
August 16, 1951	6 (Additional)	200
December 27, 1951	6	80 (Counted)§
December 18, 1952	5	188 (Counted)§
December 22–23, 1953	3	177 (Counted)§
	Aripo River	
December 24–29, 1953	5	500

* 363 *Maculatus* males released April 15, 1951; 140, April 16, 1951; total, 503. Estimated population of immediate colony into which released, 60 males.

† 478 *Maculatus* males released April 16, 1951. Estimated population into which released, 200 males.

‡ 89 *Maculatus* males released December 22, 1953. Estimated population into which released, 500 males.

§ In cases so marked, samples were caught and counted as indicated. In all other cases the population was estimated by capture or release-recapture methods and the total number of *Maculatus* males found in the population recorded.

‖ Destroyed.

[1] E. B. Ford, "Polymorphism and Taxonomy," in Julian Huxley (ed.), *New Systematics* (Oxford: Oxford University Press, 1940); *Mendelism and Evolution* ("Methuen's Monographs") (London: Methuen Co., 1949).

[2] A. W. Bellamy, "Multiple Allelomorphism and Inheritance of Color Patterns in Tettigidae," *J. Genetics*, **7**, 55–70 (1917).

[3] R. K. Nabours, "The Genetics of the Tettigidae (Grouse Locusts)," *Bibl. Genet.*, **5**, 27–104 (1929).

[4] R. K. Nabours and B. Snyder, "Partheogenesis and the Inheritance of Color Patterns in the Grouse Locust, *Telemetettix aztecus* Saussune," *Genetics*, **13**, 126–132 (1928).

[5] R. K. Nabours, I. Larson, and N. Hartwig, "Inheritance of Color Patterns in the Grouse Locust *Acrydium arenosum* Burmeister. (Tettigidae)," *Genetics*, **18**, 159–171 (1933).

[6] I. A. Rubtzov, "Phase Variation in Non-swarming Grasshoppers," *Bull. Ent. Res.*, **26**, 499–524 (1953).

[7] C. M. Good, "The Genetics of the Grouse Locust *Tettigidea parvipennis* Harris," *Trans. Kansas Acad. Sci.*, **44**, 234–237 (1941).

[8] A. de Zulueta, "La Herencia ligada al sexo en el coléoptero *Phytodecta variabilis*," *Eos*, 1, 203–229 (1925).

[9] N. W. Timofeef-Ressovsky, "Zur Analyse des Polymorphismus bei *Adalia bipunctata*," *Biol. Zentralbl.*, **69**, 130–137 (1940).

[10] A. F. Shull, "Inheritance in Lady Beetles. I. The Spotless and Spotted Elytra of *Hippodamia sinuata*," *J. Heredity*, **34**, 329–337 (1943).

[11] C. C. Tan, "Seasonal Variations in Color Patterns in *Harmonia axyridis*," *Proc. 8th Internat. Cong. Genetics* (*Hereditas*, suppl. vol.), pp. 669–670 (1949).

[12] T. Komai, H. Chino, and Y. Hosino, "Contributions to the Evolutionary Genetics of the Lady Beetle *Harmonia*. I. Geographic and Temporal Variations in the Relative Frequencies of the Elytral Pattern Types and in the Frequency of Elytral Ridge," *Genetics*, **35**, 589–601 (1950).

[13] T. Komai and Y. Hosino, "Contributions to the Evolutionary Genetics of the Lady Beetle *Harmonia*. II. Microgeographic Variations," *Genetics*, **36**, 382–390 (1951).

[14] Th. Dobzhansky, *Genetics and the Origin of Species* (New York: Columbia University Press, 1937).

[15] J. H. Gerould, "Genetics of Butterflies (*Colias*)," *Genetics*, **26**, 152 (1941).

[16] W. Hovanitz, *Polymorphism and Evolution: Symposia of the Study for Experimental Biology* (New York: Academic Press, Inc., 1953).

[17] A. C. Kinsey, "Local Populations in the Gall Wasp *Biorrhiza eburnea*," *Genetics*, **26**, 158 (1941).

[18] M. Gordon, "Speciation in Fishes," *Advances in Genetics*, Vol. 1 (New York, Academic Press, Inc., 1947).

[19] C. Diver, "Fossil Records of Mendelian Mutants," *Nature*, **124**, 183 (1929).

[20] C. Diver, "Mollusca Genetics," *Proc. 6th Internat. Cong. Genetics*, **2**, 236–237 (1932); "The Problem of Closely Related Species Living in the Same Area," in Julian Huxley (ed.), *New Systematics* (London: Oxford University Press, 1940).

[21] R. A. Fisher and C. Diver, "Crossing-over in the Land Snail, *Cepea nemoralis* L.," *Nature*, **133**, 834 (1934).

[22] A. J. Cain and P. M. Sheppard, "Selection in the Polymorphic Land Snail *Cepea nemoralis*," *Heredity*, **4**, 275–294 (1950); "The Effects of Natural Selection on Body Color in the Snail, *Cepea nemoralis*," *ibid.*, **6**, 217–231 (1952).

[23] In a sense this is the converse of the situation especially considered by Lerner (*Population Genetics and Animal Improvement* [Cambridge: At the University Press, 1950]) involving "homeostasis of genetic ratios"—the tendency of the gene ratios in a polymorphic population to return to a norm for the species after disturbance by artificial selection.

[24] J. Schmidt, "Racial Investigations. IV. The Genetic Behavior of a Secondary Sexual Character," *C. R. Lab. Carlsberg*, Vol. **14**, No. 8 (1920).

[25] L. J. Blacher, "Materials for the Study of *Lebistes reticulatus*," *Trans. Lab. Exp. Biol. Zoopark Moscow*, **3**, 151 (1927).

[26] H. B. Goodrich, N. D. Josephson, J. P. Trinkaus, and J. M. Slate, "The Cellular Expression and Genetics of Two New Genes in *Lebistes reticulatus*," *Genetics*, **29**, 584 (1944).

[27] O. Winge, "One Sided Masculine and Sex-linked Inheritance in *Lebistes reticulatus*," *C.R. Lab. Carlsberg*, Vol. **14**, No. 18 (1922); *J. Genetics*, **12**, 145 (1922); "The Location of Eighteen Genes in *Lebistes reticulatus*," *J. Genetics*, **18**, i (1927).

[28] O. Winge, "A Peculiar Mode of Inheritance and Its Cytological Explanation," *C.R. Lab. Carlsberg*, Vol. **14**, No. 17 (1921); *J. Genetics*, **12**, 137 (1922); "Crossing over between the *X*- and *Y*-Chromosome in *Lebistes*," *C.R. Lab. Carlsberg*, Vol. **14**, No. 20 (1922); *J. Genetics*, **13**, 201 (1923); "On the Occurrence of *XX* Males in *Lebistes*, with Some Remarks on Aida's So-called 'Non-disjunctional' Males in *Aplocheilus*," *J. Genetics*, **23**, 69 (1930); "The Nature of Sex Chromosomes," *Proc. 6th Internat. Cong. Genetics*, **1**, 343 (1932); "The Experimental Alteration of Sex Chromosomes into Autosomes and vice versa, as Illustrated by *Lebistes*," *C.R. Lab. Carlsberg*, sér. physiol., **21**, i (1934); O. Winge and E. Ditlevsen, "A Lethal Gene in the *Y*- Chromosome of *Lebistes*, *C.R. Lab. Carlsberg*, sér. physiol., **22**, 203 (1938); "Colour Inheritance and Sex Determination in *Lebistes*," *ibid.*, **24**, 227 (1948).

[29] C. P. Haskins and E. F. Haskins, "The Inheritance of Certain Color Patterns in Wild Populations of *Lebistes reticulatus* in Trinidad," *Evolution*, **5**, 216 (1951).

NATIONAL ACADEMY OF SCIENCES

July 1, 1954

OFFICERS

	Term expires
President—Detlev W. Bronk	June 30, 1958
Vice President—George W. Corner	June 30, 1957
Home Secretary—Alexander Wetmore	June 30, 1955
Foreign Secretary—John G. Kirkwood	June 30, 1958
Treasurer—William J. Robbins	June 30, 1956

Executive Officer
S. D. Cornell

Business Manager
G. D. Meid

COUNCIL

*Bronk, Detlev W.	(1958)	*Robbins, William J.	(1956)
*Corner, George W.	(1957)	*Rubey, W. W.	†
Daniels, Farrington	(1957)	Stanley, Wendell M.	(1955)
*Dryden, Hugh L.	(1956)	Tuve, Merle A.	(1957)
Kirkwood, John G.	(1957)	*Wetmore, Alexander	(1955)
Loeb, Robert F.	(1955)	*Wilson, Edwin B.	(1956)

MEMBERS

The number in parentheses, following year of election, indicates the Section to which the member belongs, as follows:

(1) Mathematics
(2) Astronomy
(3) Physics
(4) Engineering
(5) Chemistry
(6) Geology
(7) Botany
(8) Zoology and Anatomy
(9) Physiology and Biochemistry
(10) Pathology and Bacteriology
(11) Anthropology
(12) Psychology
(13) Geophysics

Abbot, Charles **G**reeley, 1915 (2), Smithsonian Institution, Washington 25, D. C.

Adams, Comfort **A**very, 1930 (4), 417 West Price Street, Philadelphia 44, Pennsylvania

Adams, Leason **H**eberling, 1943 (13), Carnegie Institution of Washington, 1530 P Street, N. W., Washington 5, D. C.

Adams, Roger, 1929 (5), Department of Chemistry and Chemical Engineering, University of Illinois, Urbana, Illinois

Adams, Walter Sydney, 1917 (2), Mount Wilson Observatory, Pasadena 4, California

Ahlfors, Lars Valerian, 1953 (1), Department of Mathematics, Harvard University, Cambridge 38, Massachusetts

Albert, Abraham **Adrian,** 1943 (1), Department of Mathematics, University of Chicago, Chicago 37, Illinois

* Members of the Executive Committee of the Council of the Academy.
† *Ex officio* as Chairman of the National Research Council.

Albright, Fuller, 1952 (10), Massachusetts General Hospital, Boston 14, Massachusetts

Alexander, James Waddell, 1930 (1), 29 Cleveland Lane, Princeton, New Jersey

Allee, Warder **C**lyde, 1951 (8), Department of Biology, University of Florida, Gainesville, Florida

Allen, Eugene Thomas, 1930 (6), The Irvington, 135 Pleasant Street, Arlington 74, Massachusetts

Allison, Samuel King, 1946 (3), Institute for Nuclear Studies, University of Chicago, Chicago 37, Illinois

Alvarez, Luis Walter, 1947 (3), Radiation Laboratory, University of California, Berkeley 4, California

Anderson, Carl David, 1938 (3), California Institute of Technology, Pasadena 4, California

Anderson, Edgar, 1954 (7), Missouri Botanical Garden, 2315 Tower Grove Avenue, St. Louis 10, Missouri

Anderson, Rudolph John, 1946 (9), Sterling Chemistry Laboratory, Yale University, 225 Prospect Street, New Haven 11, Connecticut

Armstrong, Charles, 1944 (10), National Institutes of Health, Bethesda 14, Maryland

Avery, Oswald Theodore, 1933 (10), Hoods Hill Road, Nashville 12, Tennessee

Babcock, Ernest Brown, 1946 (7), Department of Genetics, University of California, Berkeley 4, California

Babcock, Harold Delos, 1933 (2), Mount Wilson Observatory, Pasadena 4, California

Babcock, Horace Welcome, 1954 (2), Mount Wilson Observatory, Pasadena 4, California

Bacher, Robert Fox, 1947 (3), California Institute of Technology, Pasadena 4, California

Badger, Richard McLean, 1952 (5), Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena 4, California

Bailey, Irving **W**idmer, 1929 (7), Harvard University Herbarium, 22 Divinity Avenue, Cambridge 38, Massachusetts

Bailey, Percival, 1953 (10), Neuropsychiatric Institute, 912 South Wood Street, Chicago 12, Illinois

Bain, Edgar Collins, 1954 (4), United States Steel Corporation, Pittsburgh, Pennsylvania

Bainbridge, Kenneth Tompkins, 1946 (3), Department of Physics, Harvard University, Cambridge 38, Massachusetts

Ball, Eric Glendinning, 1948 (9), Department of Biological Chemistry, Harvard Medical School, 25 Shattuck Street, Boston 15, Massachusetts

Balls, Arnold Kent, 1954 (9), Department of Biochemistry, Purdue University, Lafayette, Indiana

Bard, Philip, 1944 (9), School of Medicine, Johns Hopkins University, 710 North Washington Street, Baltimore 5, Maryland

Bardeen, John, 1954 (3), Department of Physics, University of Illinois, Urbana, Illinois

Barker, Horace **A**lbert, 1953 (7), Department of Plant Biochemistry, 337 Biochemistry and Virus Laboratory, University of California, Berkeley 4, California

Bartelmez, George William, 1949 (8), Department of Embryology, Carnegie Institution of Washington, Wolfe and Madison Streets, Baltimore 5, Maryland

Bartlett, Paul Doughty, 1947 (5), Department of Chemistry, Harvard University, 12 Oxford Street, Cambridge 38, Massachusetts

Beach, Frank Ambrose, 1949 (12), Department of Psychology, Yale University, 333 Cedar Street, New Haven 11, Connecticut

Beadle, George Wells, 1944 (7), Division of Biology, California Institute of Technology, Pasadena 4, California

Beams, Jesse Wakefield, 1943 (3), Physical Laboratory, University of Virginia, Charlottesville, Virginia

Bell, Eric Temple, 1927 (1), 434 South Michigan Avenue, Pasadena 5, California

Benioff, Victor Hugo, 1953 (13), Seismological Laboratory, California Institute of Technology, Pasadena 2, California

Berkey, Charles Peter, 1927 (6), Department of Geology, Columbia University, New York 27, New York

Berkner, Lloyd Viel, 1948 (13), Associated Universities, Incorporated, 350 Fifth Avenue, New York 1, New York

Bethe, Hans Albrecht, 1944 (3), Laboratory of Nuclear Studies, Cornell University, Ithaca, New York

Bigelow, Henry Bryant, 1931 (8), Museum of Comparative Zoology at Harvard College, Oxford Street, Cambridge 38, Massachusetts

Birch, Albert Francis, 1950 (6), Dunbar Laboratory, Harvard University, Cambridge 38, Massachusetts

Birge, Raymond Thayer, 1932 (3), University of California, Berkeley 4, California

Bjerknes, Jacob, 1947 (13), Department of Meteorology, University of California, Los Angeles 24, California

Blackwelder, Eliot, 1936 (6), P. O. Box N, Stanford, California

Blakeslee, Albert Francis, 1929 (7), Genetics Experiment Station, Smith College, Northampton, Massachusetts

Blalock, Alfred, 1945 (10), Johns Hopkins Hospital, Baltimore 5, Maryland

Bloch, Felix, 1948 (3), Department of Physics, Stanford University, Stanford, California

Bloom, William, 1954 (8), Department of Anatomy, University of Chicago, Chicago 37, Illinois

Bochner, Salomon, 1950 (1), Department of Mathematics, Princeton University, Princeton, New Jersey

Bodine, Joseph Hall, 1953 (8), Department of Zoology, State University of Iowa, Iowa City, Iowa

Bolton, Elmer K., 1946 (5), 2310 West 11th Street, Wilmington, Delaware

Bonner, James Frederick, 1950 (7), Kerckhoff Laboratories of Biology, California Institute of Technology, Pasadena 4, California

Boring, Edwin Garrigues, 1932 (12), Memorial Hall, Harvard University, Cambridge 38, Massachusetts

Bowen, Ira Sprague, 1936 (2), Mount Wilson and Palomar Observatories, 813 Santa Barbara Street, Pasadena 4, California.

Bowen, Norman Levi, 1935 (6), Geophysical Laboratory, Carnegie Institution of Washington, 2801 Upton Street, N. W., Washington 8, D. C.

Bradbury, Norris Edwin, 1951 (3), Los Alamos Scientific Laboratory, P. O. Box 1663, Los Alamos, New Mexico

Bradley, Wilmot Hyde, 1946 (6), United States Geological Survey, Department of the Interior, Washington 25, D. C.

Bramlette, Milton Nunn, 1954 (6), Scripps Institution of Oceanography, La Jolla, California

Breit, Gregory, 1939 (3), Sloane Physics Laboratory, Yale University, New Haven 11, Connecticut

Bridgman, Percy Williams, 1918 (3), Lyman Laboratory of Physics, Harvard University, Cambridge 38, Massachusetts

Briggs, Lyman James, 1942 (4), National Bureau of Standards, Washington 25, D. C.

Brillouin, Leon, 1953 (3), 88 Central Park West, New York 23, New York

Brink, Royal Alexander, 1947 (7), Department of Genetics, University of Wisconsin, Madison 6, Wisconsin.

Brode, Robert Bigham, 1949 (3), Department of Physics, University of California, Berkeley 4, California

Brode, Wallace Reed, 1954 (5), National Bureau of Standards, Washington 25, D. C.

Bronk, Detlev Wulf, 1939 (9), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Brouwer, Dirk, 1951 (2), Yale University Observatory, Prospect and Canner Streets, New Haven 11, Connecticut

Bucher, Walter Hermann, 1938 (6), Department of Geology, Columbia University, New York 27, New York

Buckley, Oliver Ellsworth, 1937 (4), 13 Fairview Terrace, Maplewood, New Jersey

Buddington, Arthur Francis, 1943 (6), Princeton University, Princeton, New Jersey

Buerger, Martin Julian, 1953 (6), Department of Geology and Geophysics, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Burkholder, Paul Rufus, 1949 (7), Department of Bacteriology, University of Georgia, Athens, Georgia

Bush, Vannevar, 1934 (4), Carnegie Institution of Washington, 1530 P Street, N.W., Washington 5, D. C.

Byerly, Perry, 1946 (13), Department of Geological Sciences, University of California, Berkeley 4, California

Byers, Horace Robert, 1952 (13), Department of Meteorology, University of Chicago, Chicago 37, Illinois

Calvin, Melvin, 1954 (5), Department of Chemistry, University of California, Berkeley 4, California

Cannon, Paul Roberts, 1946 (10), Department of Pathology, University of Chicago, Chicago 37, Illinois

Carlson, Anton Julius, 1920 (9), University of Chicago, Chicago 37, Illinois

Carmichael, Leonard, 1943 (12), Smithsonian Institution, Washington 25, D. C.

Carter, Herbert Edmund, 1953 (9), Department of Chemistry, University of Illinois, Urbana, Illinois

Castle, William Bosworth, 1939 (10), Boston, City Hospital, Boston 18, Massachusetts

Castle, William Ernest, 1915 (8), 421 Spruce Street, Berkeley 8, California

Chance, Britton, 1954 (9), Johnson Foundation, University of Pennsylvania, Philadelphia 4, Pennsylvania

Chandler, William Henry, 1943 (7), College of Agriculture, University of California, 405 Hilgard Avenue, Los Angeles 24, California

Chaney, Ralph Works, 1947 (6), Department of Paleontology, University of California, Berkeley 4, California

Chevalley, Claude, 1952 (1), 468 Riverside Drive, New York 27, New York

Clark, William Mansfield, 1928 (9), Department of Chemistry, Johns Hopkins University, Baltimore 18, Maryland

Clarke, Hans Thacher, 1942 (9), College of Physicians and Surgeons, 630 West 168th Street, New York 32, New York

Clausen, Roy Elwood, 1951 (7), 314 Hilgard Hall, University of California, Berkeley 4, California

Cleland, Ralph Erskine, 1942 (7), Department of Botany, Indiana University, Bloomington, Indiana

Clemence, Gerald Maurice, 1952 (2), United States Naval Observatory, Washington 25, D. C.

Cleveland, Lemuel Roscoe, 1952 (8), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Cloos, Ernst, 1950 (6), Department of Geology, Johns Hopkins University, Baltimore 18, Maryland

Coble, Arthur Byron, 1924 (1), 1907 Tondolea Lane, La Canada, California

Coblentz, William Weber, 1930 (3), 2737 Macomb Street, N. W., Washington 8, D. C.

Cochrane, Edward Lull, 1945 (4), Office of Industrial and Governmental Relations, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Coggeshall, Lowell Thelwell, 1949 (10), Division of Biological Sciences, University of Chicago, Chicago 37, Illinois

Cole, Rufus, 1922 (10), Mt. Kisco, New York

Compton, Arthur Holly, 1927 (3), Washington University, St. Louis 5, Missouri

Conant, James Bryant, 1929 (5), United States High Commissioner to Germany, A.P.O. 80, Care of Postmaster, New York, New York

Condon, Edward Uhler, 1944 (3), 5 Crestwood Road, Corning, New York

Coolidge, William David, 1925 (3), 1480 Lenox Road, Schenectady 8, New York

Cope, Arthur Clay, 1947 (5), Department of Chemistry, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Cori, Carl Ferdinand, 1940 (9), School of Medicine, Washington University, Euclid Avenue and Kingshighway, St. Louis 10, Missouri

Cori, Gerty Theresa, 1948 (9), School of Medicine, Washington University, Euclid Avenue and Kingshighway, St. Louis 10, Missouri

Corner, George Washington, 1940 (8), Department of Embryology, Carnegie Institution of Washington, Wolfe and Madison Streets, Baltimore 5, Maryland

Couch, John Nathaniel, 1943 (7), University of North Carolina, Chapel Hill, North Carolina

Craig, Lyman Creighton, 1950 (5), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Curme, George Oliver, Jr., 1944 (4), Union Carbide and Carbon Corporation, 30 East 42 Street, New York 17, New York

Daly, Reginald Aldworth, 1925 (6), 23 Hawthorn Street, Cambridge 38, Massachusetts

Danforth, Charles Haskell, 1942 (8), Department of Anatomy, Stanford University, Stanford, California

Daniels, Farrington, 1947 (5), Department of Chemistry, University of Wisconsin, Madison 6, Wisconsin

Davis, Bergen, 1929 (3), 44 Morningside Drive, New York 25, New York

Davis, Hallowell, 1948 (9), Central Institute for the Deaf, 818 South Kingshighway, St. Louis 10, Missouri

Davisson, Clinton Joseph, 1929 (3), 2605 Jefferson Park Circle, Charlottesville, Virginia

Day, Arthur Louis, 1911 (6), 9113 Old Georgetown Road, Bethesda 14, Maryland

Debye, Peter, (1931) 1947* (5), Baker Laboratory, Cornell University, Ithaca, New York

DeGolyer, Everette Lee, 1951 (6), 5625 Daniels Street, Dallas, Texas

Delbrück, Max, 1949 (7), Kerckhoff Laboratories of Biology, California Institute of Technology, Pasadena 4, California

Demerec, Milislav, 1946 (8), Department of Genetics, Carnegie Institution of Washington, Cold Spring Harbor, Long Island, New York

Den Hartog, Jacob Pieter, 1953 (4), Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Dennison, David Mathias, 1953 (3), Randall Laboratory of Physics, University of Michigan, Ann Arbor, Michigan

Detwiler, Samuel Randall, 1932 (8), College of Physicians and Surgeons, 630 West 168th Street, New York 32, New York

Dobzhansky, Theodosius, 1943 (8), Department of Zoology, Columbia University, New York 27, New York

Dochez, Alphonse Raymond, 1933 (10), Columbia University, Presbyterian Hospital, 620 West 168th Street, New York 32, New York

Dodge, Bernard Ogilvie, 1933 (7), New York Botanical Garden, Bronx Park (Fordham Station), New York 58, New York

Doisy, Edward Adelbert, 1938 (9), St. Louis University School of Medicine, 1402 South Grand Boulevard, St. Louis 4, Missouri

Douglas, Jesse, 1946 (1), Butler Hall, 400 West 119th Street, New York 27, New York

Dragstedt, Lester Reynold, 1950 (10), Department of Surgery, University of Chicago, 950 East 59th Street, Chicago 37, Illinois

Dryden, Hugh Latimer, 1944 (4), National Advisory Committee for Aeronautics, 1512 H Street, N. W., Washington 25, D. C.

DuBois, Eugene Floyd, 1933 (9), Cornell University Medical College, 1300 York Avenue, New York 21, New York

* Elected a foreign associate in 1931; became a naturalized citizen in 1946 and a member of the Academy in 1947.

Dubos, René Jules, 1941 (10), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

DuBridge, Lee Alvin, 1943 (3), California Institute of Technology, Pasadena 4, California

Duggar, Benjamin Minge, 1927 (7), Lederle Laboratories Division, American Cyanamid Company, Pearl River, New York

DuMond, Jesse William Monroe, 1953 (3), Department of Physics, California Institute of Technology, Pasadena 4, California

Dunbar, Carl Owen, 1944 (6), Peabody Museum, Yale University, New Haven 11, Connecticut

Dunn, Leslie Clarence, 1943 (8), Columbia University, New York 27, New York

Dunning, John Ray, 1948 (3), 301 Engineering Building, Columbia University, New York 27, New York

Durand, William Frederick, 1917 (4), Mohawk Hotel, 379 Washington Avenue, Brooklyn, New York

du Vigneaud, Vincent, 1944 (9), Department of Biochemistry, Cornell University Medical College, 1300 York Avenue, New York 21, New York

Eckart, Carl, 1953 (13), Scripps Institution of Oceanography, La Jolla, California

Edsall, John Tileston, 1951 (5), The Biological Laboratories, 16 Divinity Avenue, Cambridge 38, Massachusetts

Einstein, Albert (1922) 1942† (3), The Institute for Advanced Study, Princeton, New Jersey

Eisenhart, Luther Pfahler, 1922 (1), 25 Alexander Street, Princeton, New Jersey

Elderfield, Robert Cooley, 1949 (5), Department of Chemistry, University of Michigan, Ann Arbor, Michigan

Elvehjem, Conrad Arnold, 1942 (9), Department of Biochemistry, University of Wisconsin, Madison 6, Wisconsin

Emerson, Robert, 1953 (7), Department of Botany, University of Illinois, Urbana, Illinois

Enders, John Franklin, 1953 (10), The Children's Hospital, 300 Longwood Avenue, Boston 15, Massachusetts

Epstein, Paul Sophus, 1930 (3), 1484 Oakdale Street, Pasadena 4, California

Erlanger, Joseph, 1922 (9), 5127 Waterman Boulevard, St. Louis 8, Missouri

Evans, Griffith Conrad, 1933 (1), Department of Mathematics, University of California, Berkeley 4, California

Evans, Herbert McLean, 1927 (9), Institute of Experimental Biology, University of California, Berkeley 4, California

Ewing, William Maurice, 1948 (13), Columbia University, Lamont Geological Observatory, Torrey Cliff, Palisades, New York

Eyring, Henry, 1945 (5), Graduate School, University of Utah, Salt Lake City, Utah

Fenn, Wallace Osgood, 1943 (9), School of Medicine and Dentistry, University of Rochester, 260 Crittenden Boulevard, Rochester 7, New York

Fermi, Enrico, 1945 (3), Institute for Nuclear Studies, University of Chicago, Chicago 37, Illinois

† Elected a foreign associate in 1922; became a naturalized citizen in 1941 and a member of the Academy in 1942.

Feynman, Richard Phillips, 1954 (3), Norman Bridge Laboratory of Physics, California Institute of Technology, Pasadena 4, California

Fieser, Louis Frederick, 1940 (5), Harvard University, Cambridge 38, Massachusetts

Fischer, Hermann Otto Laurenz, 1954 (9), Department of Biochemistry, University of California, Berkeley 4, California

Fisk, James Brown, 1954 (4), Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey

Fleming, John Adam, 1938 (13), Apt. 6A, Shelby Court, 4th Street and Shelby Avenue, Nashville 6, Tennessee

Fletcher, Harvey, 1935 (4), College of Physical and Engineering Sciences, Brigham Young University, Provo, Utah

Flory, Paul John, 1953 (5), Department of Chemistry, Cornell University, Ithaca, New York

Folkers, Karl August, 1948 (5), Organic and Biological Chemical Research Division, Merck & Co., Inc., Rahway, New Jersey

Foote, Paul Darwin, 1943 (4), P. O. Drawer 2038, Pittsburgh 30, Pennsylvania

Forbes, Alexander, 1936 (9), Harland Street, Milton, Massachusetts

Francis, Thomas, Jr., 1948 (10), Department of Epidemiology, School of Public Health, University of Michigan, Ann Arbor, Michigan

Franck, James, 1944 (3), Institute of Radiobiology and Biophysics, University of Chicago, Chicago 37, Illinois

Fred, Edwin Broun, 1931 (7), University of Wisconsin, Madison 6, Wisconsin

Fruton, Joseph Stewart, 1952 (9), Department of Biochemistry, Yale University School of Medicine, 333 Cedar Street, New Haven 11, Connecticut

Fuoss, Raymond Matthew, 1951 (5), Sterling Chemistry Laboratory, Yale University, 225 Prospect Street, New Haven 11, Connecticut

Fuson, Reynold Clayton, 1944 (5), 263 Noyes Laboratory, University of Illinois, Urbana, Illinois

Gamble, James Lawder, 1945 (10), 33 Edge Hill Road, Brookline, Massachusetts

Gamow, George, 1953 (3), Department of Physics, George Washington University, Washington, D. C.

Gasser, Herbert Spencer, 1934 (9), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Gesell, Arnold, 1947 (12), Gesell Institute of Child Development, 310 Prospect Street, New Haven 11, Connecticut

Giauque, William Francis, 1936 (5), Department of Chemistry and Chemical Engineering, University of California, Berkeley 4, California

Gibbs, William Francis, 1949 (4), One Broadway, New York, New York

Gilliland, Edwin Richard, 1948 (4), Department of Chemical Engineering, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Gilluly, James, 1947 (6), United States Geological Survey, Denver Federal Center, Denver 14, Colorado

Gilman, Henry, 1945 (5), Department of Chemistry, Iowa State College, Ames, Iowa

Goddard, David Rockwell, 1950 (7), Botanical Laboratory, University of Pennsylvania, Philadelphia 4, Pennsylvania

Goldschmidt, Richard Benedikt, 1947 (8), Department of Zoology, University of California, Berkeley 4, California

Goodpasture, Ernest William, 1937 (10), Vanderbilt University, Nashville 4, Tennessee

Goudsmit, Samuel Abraham, 1947 (3), Department of Physics, Brookhaven National Laboratory, Upton, Long Island, New York

Graham, Clarence **Henry,** 1946 (12), Department of Psychology, Columbia University, New York 27, New York

Graham, Evarts Ambrose, 1941 (10), Barnes Hospital, 600 South Kingshighway, St. Louis 10, Missouri

Greenewalt, Crawford Hallock, 1952 (4), E. I. du Pont de Nemours and Company, Incorporated, Wilmington 98, Delaware

Gregory, William King, 1927 (6), American Museum of Natural History, 77th Street and Central Park West, New York 24, New York

Griggs, David Tressel, 1952 (13), Institute of Geophysics, University of California, Los Angeles 24, California

Guilford, Joy Paul, 1954 (12), Department of Psychology, University of Southern California, Los Angeles 7, California

Gunn, Ross, 1951 (13), United States Weather Bureau, Department of Commerce, Washington 25, D. C.

Gutenberg, Beno, 1945 (13), Seismological Laboratory, 220 North San Rafael Avenue, Pasadena 2, California

Hamburger, Viktor, 1953 (8), Department of Zoology, Washington University, St. Louis 5, Missouri

Hammett, Louis Plack, 1943 (5), Department of Chemistry, Columbia University, New York 27, New York

Harlow, Harry F., 1951 (12), Department of Psychology, University of Wisconsin, Madison 6, Wisconsin

Harned, Herbert Spencer, 1950 (5), Sterling Chemistry Laboratory, Yale University, 225 Prospect Street, New Haven 11, Connecticut

Harrison, Ross Granville, 1913 (8), Osborn Zoological Laboratory, Yale University, New Haven 11, Connecticut

Hartline, Haldan **Keffer,** 1948 (9), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Hartman, Carl Gottfried, 1937 (8), Ortho Research Foundation, Raritan, New Jersey

Harvey, Edmund **Newton,** 1934 (8), Guyot Hall, Princeton University, Princeton, New Jersey

Hastings, Albert **Baird,** 1939 (9), Department of Biological Chemistry, Harvard Medical School, 25 Shattuck Street, Boston 15, Massachusetts

Heidelberger, Michael, 1942 (10), College of Physicians and Surgeons, 620 West 168th Street, New York 32, New York

Hendricks, Sterling Brown, 1952 (5), Plant Industry Station, Beltsville, Maryland

Herrick, Charles **Judson,** 1918 (8), 236 Morningside Drive, S. E., Grand Rapids 6, Michigan

Hess, Harry Hammond, 1952 (6), Department of Geology, Princeton University, Princeton, New Jersey

Hewett, Donnel Foster, 1937 (6), 1460 Rose Villa Street, Pasadena 4, California

Hildebrand, Joel Henry, 1929 (5), Department of Chemistry, University of California, Berkeley 4, California

Hilgard, Ernest Ropiequet, 1948 (12), Graduate Division, Stanford University, Stanford, California

Hille, Carl Einar, 1953 (1), Department of Mathematics, Yale University, New Haven 11, Connecticut

Hirschfelder, Joseph Oakland, 1953 (5), University of Wisconsin, P. O. Box 2127, Madison 6, Wisconsin

Hisaw, Frederick Lee, 1947 (8), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Hoover, Herbert Clark, 1922 (4), Waldorf Astoria Towers, New York, New York

Horsfall, Frank Lappin, Jr., 1948 (10), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Horsfall, James Gordon, 1953 (7), Connecticut Agricultural Experiment Station, New Haven 4, Connecticut

Houston, William Vermillion, 1943 (3), The Rice Institute, Houston, Texas

Hubbs, Carl Leavitt, 1952 (8), Scripps Institution of Oceanography, La Jolla, California

Huggins, Charles Brenton, 1949 (10), Department of Surgery, University of Chicago, 950 East 59th Street, Chicago 37, Illinois

Hull, Albert Wallace, 1929 (3), General Electric Research Laboratory, The Knolls, Schenectady, New York

Hunsaker, Jerome Clark, 1935 (4), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Hunter, Walter Samuel, 1935 (12), Brown University, Providence 12, Rhode Island

Hutchinson, George Evelyn, 1950 (8), Osborn Zoological Laboratory, Yale University, New Haven 11, Connecticut

Irwin, Malcolm Robert, 1950 (8), Department of Genetics, University of Wisconsin, Madison 6, Wisconsin

Iselin, Columbus O'Donnell, 1951 (13), Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

Jacobs, Merkel Henry, 1939 (8), School of Medicine, University of Pennsylvania, Philadelphia 4, Pennsylvania

Jacobs, Walter Abraham, 1932 (5), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Jacobson, Nathan, 1954 (1), Department of Mathematics, Yale University, New Haven 11, Connecticut

Jeffries, Zay, 1939 (4), General Electric Company, 1 Plastics Avenue, Pittsfield, Massachusetts

Johnson, John Raven, 1948 (5), Department of Chemistry, Cornell University, Ithaca, New York

Johnson, William Summer, 1952 (5), Department of Chemistry, University of Wisconsin, Madison 6, Wisconsin

Jones, Donald Forsha, 1939 (7), Department of Genetics, Connecticut Agricultural Experiment Station, P. O. Box 1106, New Haven 4, Connecticut

Joy, Alfred Harrison, 1944 (2), Mount Wilson Observatory, Pasadena 4, California

Kasner, Edward, 1917 (1), 430 West 116th Street, New York 27, New York

Kaufmann, Berwind Petersen, 1952 (7), Department of Genetics, Carnegie Institution of Washington, Cold Spring Harbor, Long Island, New York

Kelley, Walter Pearson, 1943 (6), 120 Hilgard Hall, University of California, Berkeley 4, California

Kellogg, Arthur **Remington,** 1951 (8), United States National Museum, Smithsonian Institution, Washington 25, D. C.

Kelly, Mervin J., 1945 (4), Bell Telephone Laboratories, Incorporated, 463 West Street, New York 14, New York

Kemble, Edwin Crawford, 1931 (3), Physics Laboratories, Harvard University, Cambridge 38, Massachusetts

Kendall, Edward Calvin, 1950 (9), 3 Queenston Place, Princeton, New Jersey

Kent, Robert Harrington, 1951 (3), Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland

Kerst, Donald William, 1951 (3), Department of Physics, University of Illinois, Urbana, Illinois

Kettering, Charles Franklin, 1928 (4), General Motors Corporation Research Laboratories, Box 188, North End Station, Detroit 2, Michigan

Keyes, Frederick George, 1930 (5), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Kharasch, Morris Selig, 1946 (5), Department of Chemistry, University of Chicago, Chicago 37, Illinois

Kidder, Alfred Vincent, 1936 (11), 41 Holden Street, Cambridge 38, Massachusetts

Kimball, George Elbert, 1954 (5), Department of Chemistry, Columbia University, New York 27, New York

King, Arthur Scott, 1941 (2), Mount Wilson Observatory, Pasadena 4, California

King, Charles Glen, 1951 (9), The Nutrition Foundation, Incorporated, Chrysler Building, New York 17, New York

Kirkwood, John Gamble, 1942 (5), Sterling Chemistry Laboratory, Yale University, 225 Prospect Street, New Haven 11, Connecticut

Kistiakowsky, George Bogdan, 1939 (5), Department of Chemistry, Harvard University, 12 Oxford Street, Cambridge 38, Massachusetts

Kluckhohn, Clyde Kay Maben, 1952 (11), 50 Fresh Pond Parkway, Cambridge 38, Massachusetts

Knopf, Adolph, 1931 (6), Department of Geology, Stanford University, Stanford, California

Köhler, Wolfgang, 1947 (12), Swarthmore College, Swarthmore, Pennsylvania

Kraus, Charles August, 1925 (5), Brown University, Providence 12, Rhode Island

Kroeber, Alfred **L.,** 1928 (11), Department of Anthropology, University of California, Berkeley 4, California

Kuiper, Gerard Peter, 1950 (2), Yerkes Observatory, University of Chicago, Williams Bay, Wisconsin

Kunkel, Louis Otto, 1932 (7), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Lamb, Willis Eugene, **Jr.,** 1954 (3), Department of Physics, Stanford University, Stanford, California

Lambert, Walter Davis, 1949 (13), P. O. Box 687, Canaan, Connecticut

LaMer, Victor Kuhn, 1945 (5), Department of Chemistry, Columbia University, New York 27, New York

Land, Edwin Herbert, 1953 (3), Polaroid Corporation, Cambridge 39, Massachusetts

Landis, Eugene Markley, 1954 (9), Department of Physiology, Harvard Medical School, 25 Shattuck Street, Boston 15, Massachusetts

Langmuir, Irving, 1918 (5), Research Laboratory, General Electric Company, P. O. Box 1088, Schenectady, New York

Larsen, Esper S., Jr., 1944 (6), Apt. 502 H, 3930 Connecticut Avenue, N. W., Washington 8, D. C.

Lashley, Karl Spencer, 1930 (12), Yerkes Laboratories of Primate Biology, Incorporated, Orange Park, Florida

Latimer, Wendell Mitchell, 1940 (5), Gilman Hall, University of California, Berkeley 4, California

Lauritsen, Charles Christian, 1941 (3), California Institute of Technology, Pasadena 4, California

Lawrence, Ernest Orlando, 1934 (3), Radiation Laboratory, University of California, Berkeley 4, California

Lefschetz, Solomon, 1925 (1), Fine Hall, Princeton University, 129 Broadmead Street, Princeton, New Jersey

Leith, Charles Kenneth, 1920 (6), Sheraton Park Hotel, Washington 8, D. C.

Lewis, Warren Harmon, 1936 (8), The Wistar Institute of Anatomy and Biology, Philadelphia 4, Pennsylvania

Lewis, Warren Kendall, 1938 (4), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Libby, Willard Frank, 1950 (5), Institute for Nuclear Studies, University of Chicago, Chicago 37, Illinois

Lind, Samuel Colville, 1930 (5), P. O. Box P, Oak Ridge, Tennessee

Lindsley, Donald Benjamin, 1952 (12), Department of Psychology, University of California, Los Angeles 24, California

Link, Karl Paul, 1946 (9), Department of Biochemistry, University of Wisconsin, Madison 6, Wisconsin

Lipmann, Fritz Albert, 1950 (9), Biochemical Research Laboratory, Massachusetts General Hospital, Boston 14, Massachusetts

Little, Clarence Cook, 1945 (10), Roscoe B. Jackson Memorial Laboratory, Bar Harbor, Maine

Lloyd, David Pierce Caradoc, 1953 (9), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Loeb, Leo, 1937 (10), 40 Crestwood Drive, St. Louis 5, Missouri

Loeb, Robert Frederick, 1946 (9), Columbia University College of Physicians and Surgeons, 620 West 168th Street, New York 32, New York

Long, Cyril Norman Hugh, 1948 (9), Yale University School of Medicine, 333 Cedar Street, New Haven 11, Connecticut

Long, Esmond Ray, 1946 (10), Henry Phipps Institute, 7th and Lombard Streets, Philadelphia 47, Pennsylvania

Longsworth, Lewis Gibson, 1947 (5), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Longwell, Chester Ray, 1935 (6), Kirtland Hall, Yale University, New Haven 11, Connecticut

Loomis, Alfred Lee, 1941 (4), The Loomis Institute for Scientific Research, Incorporated, Room 2420, 14 Wall Street, New York 5, New York

Loomis, Francis Wheeler, 1949 (3), 804 West Illinois Street, Urbana, Illinois

Lorente de Nó, Rafael, 1950 (9), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Lothrop, Samuel Kirkland, 1951 (11), Peabody Museum, Harvard University, Cambridge 38, Massachusetts

Lovering, Thomas Seward, 1949 (6), 8001 West 17th Avenue, Lakewood, Colorado

Lowie, Robert Harry, 1931 (11), Department of Anthropology, University of California, Berkeley 4, California

Lyman, Theodore, 1917 (3), Research Laboratory of Physics, Harvard University, Cambridge 38, Massachusetts

Macelwane, James Bernard, S.J., 1944 (13), 3621 Olive Street, St. Louis 8, Missouri

MacInnes, Duncan Arthur, 1937 (5), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Mac Lane, Saunders, 1949 (1), Department of Mathematics, University of Chicago, Chicago 37, Illinois

Mangelsdorf, Paul Christoph, 1945 (7), Botanical Museum, Harvard University, Cambridge 38, Massachusetts

Mann, Frank Charles, 1950 (9), 605 11th Street, S. W., Rochester, Minnesota

Marshall, Eli Kennerly, Jr., 1943 (9), School of Medicine, Johns Hopkins University, 710 North Washington Street, Baltimore 5, Maryland

Marvel, Carl Shipp, 1938 (5), Department of Chemistry, University of Illinois, Urbana, Illinois

Mason, Max, 1923 (3), 1035 Harvard Avenue, Claremont, California

Maxcy, Kenneth Fuller, 1950 (10), School of Hygiene and Public Health, Johns Hopkins University, 615 North Wolfe Street, Baltimore 5, Maryland

Mayall, Nicholas Ulrich, 1949 (2), Lick Observatory, Mount Hamilton, California

Mayer, Joseph Edward, 1946 (5), Institute for Nuclear Studies, University of Chicago, Chicago 37, Illinois

Maynard, Leonard Amby, 1944 (9), School of Nutrition, Cornell University, Ithaca, New York

Mayr, Ernst, 1954 (8), Museum of Comparative Zoology at Harvard College, Oxford Street, Cambridge 38, Massachusetts

McClintock, Barbara, 1944 (7), Carnegie Institution of Washington, Cold Spring Harbor, Long Island, New York

McCollum, Elmer Verner, 1920 (9), Gilman Hall, Johns Hopkins University, Baltimore 18, Maryland

McElvain, Samuel Marion, 1949 (5), Department of Chemistry, University of Wisconsin, Madison 6, Wisconsin

McMaster, Philip Duryeé, 1952 (10), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

McMillan, Edwin Mattison, 1947 (3), Radiation Laboratory, University of California, Berkeley 4, California

McShane, Edward James, 1948 (1), Department of Mathematics, University of Virginia, Charlottesville, Virginia

Mead, Warren Judson, 1939 (6), 88 Rutledge Road, Belmont 78, Massachusetts

Meek, Walter Joseph, 1947 (9), Department of Physiology, University of Wisconsin, Madison 6, Wisconsin

Mees, Charles Edward Kenneth, 1950 (5), Eastman Kodak Company, Kodak Park Works, Rochester 4, New York

Meggers, William Frederick, 1954 (3), National Bureau of Standards, Washington 25, D. C.

Mendenhall, Walter Curran, 1932 (6), 9 East Lenox Street, Chevy Chase 15, Maryland

Menzel, Donald Howard, 1948 (2), Harvard College Observatory, Cambridge 38, Massachusetts

Merica, Paul Dyer, 1942 (4), 67 Wall Street, New York 5, New York

Merrill, Paul Willard, 1929 (2), 813 Santa Barbara Street, Pasadena 4, California

Metz, Charles William, 1948 (8), Department of Zoology, University of Pennsylvania, Philadelphia 4, Pennsylvania

Meyer, Karl Friederich, 1940 (10), George Williams Hooper Foundation, University of California Medical Center, San Francisco 22, California

Miles, Walter Richard, 1933 (12), Yale University School of Medicine, 333 Cedar Street, New Haven 11, Connecticut

Mirsky, Alfred Ezra, 1954 (8), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Mitchell, Samuel Alfred, 1933 (2), Leander McCormick Observatory, University Station, Charlottesville, Virginia

Moore, Carl Richard, 1944 (8), University of Chicago, Chicago 37, Illinois

Moore, Robert Lee, 1931 (1), University of Texas, Austin 12, Texas

Morse, Harold Marston, 1932 (1), The Institute for Advanced Study, Princeton, New Jersey

Muller, Hermann Joseph, 1931 (8), Zoology Department, Indiana University, Bloomington, Indiana. June 1, 1954 to December 31, 1954: 5203 Pilikai Street, Honolulu, T. H.

Mulliken, Robert Sanderson, 1936 (3), Laboratory of Molecular Structure and Spectra, Department of Physics, University of Chicago, Chicago 37, Illinois

Murnaghan, Francis Dominic, 1942 (1), 6202 Sycamore Road, Baltimore 12, Maryland

Murphree, Eger Vaughan, 1950 (4), Standard Oil Development Company, 15 West 51st Street, New York 19, New York

Nicholas, John Spangler, 1949 (8), Osborn Zoological Laboratory, Yale University, New Haven 11, Connecticut

Nicholson, Seth Barnes, 1937 (2), Mount Wilson Observatory, Pasadena 4, California

Niemann, Carl George, 1952 (5), Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena 4, California

Nier, Alfred Otto C., 1950 (3), Department of Physics, University of Minnesota, Minneapolis 14, Minnesota

Nissen, Henry W., 1953 (12), Yerkes Laboratories of Primate Biology, Incorporated, Orange Park, Florida

Nolan, Thomas Brennan, 1951 (6), United States Geological Survey, Department of the Interior, Washington 25, D. C.

Northrop, John Howard, 1934 (9), Department of Bacteriology, University of California, Berkeley 4, California

Noyes, William Albert, Jr., 1943 (5), Department of Chemistry, University of Rochester, Rochester 3, New York

O'Brien, Brian, 1954 (3), American Optical Company, Southbridge, Massachusetts

Oncley, John Lawrence, 1947 (9), Department of Biophysical Chemistry, Harvard University, 25 Shattuck Street, Boston 15, Massachusetts

Onsager, Lars, 1947 (5), Sterling Chemistry Laboratory, Yale University, 225 Prospect Street, New Haven 11, Connecticut

Opie, Eugene Lindsay, 1923 (10), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Oppenheimer, J. Robert, 1941 (3), The Institute for Advanced Study, Princeton, New Jersey

Osterhout, Winthrop John Vanleuven, 1919 (7), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Painter, Theophilus Shickel, 1938 (8), University of Texas, Austin 12, Texas

Panofsky, Wolfgang K. H., 1954 (3), Department of Physics, Stanford University, Stanford, California

Patterson, John Thomas, 1941 (8), Department of Zoology, University of Texas Austin 12, Texas

Paul, John Rodman, 1945 (10), Yale University School of Medicine, 333 Cedar Street, New Haven 11, Connecticut

Pauling, Linus, 1933 (5), Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena 4, California

Pegram, George Braxton, 1949 (3), Columbia University, New York 27, New York

Pekeris, Chaim Leib, 1952 (13), Department of Applied Mathematics, The Weizmann Institute of Science, Rehovot, Israel

Peters, John Punnett, 1947 (9), Department of Internal Medicine, Yale University School of Medicine, 333 Cedar Street, New Haven 11, Connecticut

Petrunkevitch, Alexander, 1954 (8), Osborn Zoological Laboratory, Yale University, New Haven 11, Connecticut

Pierce, George Washington, 1920 (3), 7 Berkeley Place, Cambridge 38, Massachusetts

Piggot, Charles Snowden, 1946 (13), 2200 Decatur Place, N. W., Washington 8, D. C.

Pillsbury, Walter Bowers, 1925 (12), Haven Hall, University of Michigan, Ann Arbor, Michigan

Pitzer, Kenneth Sanborn, 1949 (5), Department of Chemistry and Chemical Engineering, University of California, Berkeley 4, California

Purcell, Edward Mills, 1951 (3), Lyman Laboratory of Physics, Harvard University, Cambridge 38, Massachusetts.

Rabi, Isidor Isaac, 1940 (3), Department of Physics, Columbia University, New York 27, New York

Ramsey, Norman Foster, 1952 (3), Lyman Laboratory of Physics, Harvard University, Cambridge 38, Massachusetts

Raper, Kenneth Bryan, 1949 (7), Department of Bacteriology, University of Wisconsin, Madison 6, Wisconsin

Raymond, Arthur Emmons, 1950 (4), Douglas Aircraft Company, Incorporated, 3000 Ocean Park Boulevard, Santa Monica, California

Reeside, John Bernard, **Jr.,** 1945 (6), Box 177, Hyattsville, Maryland

Reichelderfer, Francis Wilton, 1945 (13), United States Weather Bureau, Department of Commerce, Washington 25, D. C.

Rhoades, Marcus **Morton,** 1946 (7), Department of Botany, University of Illinois, Urbana, Illinois

Rich, Arnold Rice, 1954 (10), Johns Hopkins Hospital, Baltimore 5, Maryland

Richards, Alfred Newton, 1927 (9), School of Medicine, University of Pennsylvania, Philadelphia 4, Pennsylvania

Richter, Curt Paul, 1948 (12), Johns Hopkins Hospital, Baltimore 5, Maryland

Riddle, Oscar, 1939 (8), Route 4, Plant City, Florida

Riker, Albert Joyce, 1951 (7), Department of Plant Pathology, University of Wisconsin, Madison 6, Wisconsin

Rittenberg, David, 1953 (9), College of Physicians and Surgeons, 630 West 168th Street, New York 32, New York

Rivers, Thomas Milton, 1934 (10), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Robbins, William Jacob, 1940 (7), New York Botanical Garden, Bronx Park (Fordham Station), New York 58, New York

Robertson, Howard Percy, 1951 (3), Office Chief of Staff, SHAPE, A. P. O. 55, Care of Postmaster, New York, New York

Robertson, Oswald Hope, 1943 (10), Department of Biology, Stanford University, Stanford, California

Rodebush, Worth Huff, 1938 (5), University of Illinois, Urbana, Illinois

Romer, Alfred Sherwood, 1944 (8), Museum of Comparative Zoology at Harvard College, Oxford Street, Cambridge 38, Massachusetts

Rose, William Cumming, 1936 (9), University of Illinois, Urbana, Illinois

Ross, Frank Elmore, 1930 (2), 1100 Mt. Lowe Drive, Altadena, California

Rossby, Carl-Gustaf Arvid, 1943 (13), Institute of Meteorology, University of Stockholm, Stockholm, Sweden

Rossi, Bruno Benedetto, 1950 (3), Department of Physics, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Rossini, Frederick Dominic, 1951 (5), Department of Chemistry, Carnegie Institute of Technology, Pittsburgh 13, Pennsylvania

Rous, Francis **Peyton,** 1927 (10), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Rubey, William Walden, 1945 (6), United States Geological Survey, Department of the Interior, Washington 25, D. C.

Ruedemann, Rudolf, 1928 (6), 161 Dana Avenue, Albany 3, New York

Russell, Henry Norris, 1918 (2), Princeton University, Princeton, New Jersey

Sabin, Albert Bruce, 1951 (10), The Children's Hospital Research Foundation, Elland Avenue and Bethesda, Cincinnati 29, Ohio

Saunders, Frederick Albert, 1925 (3), South Hadley, Massachusetts

Savage, John Lucian, 1949 (4), 1651 Dahlia Street, Denver 7, Colorado

Sax, Karl, 1941 (7), Arnold Arboretum, Jamaica Plain 30, Massachusetts

Scatchard, George, 1946 (5), Department of Chemistry, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Schairer, John Frank, 1953 (6), Geophysical Laboratory, Carnegie Institution of Washington, 2801 Upton Street, N. W., Washington 8, D. C.

Schlesinger, Hermann Irving, 1948 (5), Department of Chemistry, University of Chicago, Chicago 37, Illinois

Schmidt, Carl Frederic, 1949 (9), Laboratory of Pharmacology, School of Medicine, University of Pennsylvania, Philadelphia 4, Pennsylvania

Schmitt, Francis Otto, 1948 (8), Department of Biology, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Schrader, Franz, 1951 (8), Department of Zoology, Columbia University, New York 27, New York

Schultz, Adolph Hans, 1939 (11), Anthropologisches Institut, Sempersteig 3, Zurich, Switzerland

Schwinger, Julian, 1949 (3), Department of Physics, Harvard University, Cambridge 38, Massachusetts

Seaborg, Glenn Theodore, 1948 (5), Department of Chemistry, University of California, Berkeley 4, California

Seares, Frederick Hanley, 1919 (2), 351 Palmetto Drive, Pasadena 2, California

Segre, Emilio, 1952 (3), Department of Physics, University of California, Berkeley 4, California

Seitz, Frederick, 1951 (3), Department of Physics, University of Illinois, Urbana, Illinois

Serber, Robert, 1952 (3), Department of Physics, Columbia University, New York 27, New York

Shaffer, Philip Anderson, 1928 (9), School of Medicine, Washington University, Euclid Avenue and Kingshighway, St. Louis 10, Missouri

Shapiro, Harry Lionel, 1949 (11), American Museum of Natural History, Central Park West at 79th Street, New York 24, New York

Shapley, Harlow, 1924 (2), Harvard College Observatory, Cambridge 38, Massachusetts

Shedlovsky, Theodore, 1953 (5), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Sherman, Henry Clapp, 1933 (9), Columbia University, New York 27, New York

Shockley, William, 1951 (4), Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey

Shope, Richard Edwin, 1940 (10), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Simpson, George Gaylord, 1941 (6), American Museum of Natural History, 77th Street and Central Park West, New York 24, New York

Sinnott, Edmund Ware, 1936 (7), Hall of Graduate Studies, Yale University, New Haven 11, Connecticut

Skinner, Burrhus Frederic, 1950 (12), Memorial Hall, Harvard University, Cambridge 38, Massachusetts

Slater, John Clarke, 1932 (3), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Slepian, Joseph, 1941 (4), Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania

Slichter, Louis Byrne, 1944 (13), Institute of Geophysics, University of California, Los Angeles 24, California

Slipher, Vesto Melvin, 1921 (2), Lowell Observatory, Flagstaff, Arizona

Small, Lyndon Frederick, 1941 (5), National Institutes of Health, Bethesda 14, Maryland

Smith, Gilbert Morgan, 1948 (7), Department of Biology, Stanford University, Stanford, California

Smith, Homer William, 1945 (9), 477 First Avenue, New York 16, New York

Smith, Lee Irvin, 1944 (5), School of Chemistry, University of Minnesota, Minneapolis 14, Minnesota

Smith, Paul Althaus, 1947 (1), Department of Mathematics, Columbia University, New York 27, New York

Smith, Philip Edward, 1939 (8), College of Physicians and Surgeons, 630 West 168th Street, New York 32, New York

Soderberg, Carl Richard, 1947 (4), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Sonneborn, Tracy Morton, 1946 (8), 109 Wylie Hall, Indiana University, Bloomington, Indiana

Spedding, Frank Harold, 1952 (5), Box 14-A, Station A, Institute for Atomic Research, Iowa State College, Ames, Iowa

Spier, Leslie, 1946 (11), P. O. Box 880, Santa Cruz, California. February 1 to June 1, annually: University of New Mexico, Albuquerque, New Mexico

Spitzer, Lyman, Jr., 1952 (2), Princeton University Observatory, 14 Prospect Avenue, Princeton, New Jersey

Stadie, William Christopher, 1945 (9), 821 Maloney Clinic, 36th and Spruce Streets, Philadelphia 4, Pennsylvania

Stakman, Elvin Charles, 1934 (7), Institute of Agriculture, University of Minnesota, St. Paul 1, Minnesota

Stanley, Wendell Meredith, 1941 (9), Biochemistry and Virus Laboratory, University of California, Berkeley 4, California

Stebbins, George Ledyard, Jr., 1952 (7), Department of Genetics, University of California, Davis, California

Stebbins, Joel, 1920 (2), Lick Observatory, Mt. Hamilton, California

Stern, Curt, 1948 (8), Department of Zoology, University of California, Berkeley 4, California

Stern, Otto, 1945 (3), 759 Cragmont Avenue, Berkeley 8, California

Stevens, Stanley Smith, 1946 (12), Memorial Hall, Harvard University, Cambridge 38, Massachusetts

Steward, Julian Haynes, 1954 (11), Department of Sociology and Anthropology, University of Illinois, Urbana, Illinois

Stewart, George Walter, 1938 (3), Hall of Physics, State University of Iowa, Iowa City, Iowa

Stone, Calvin Perry, 1943 (12), Stanford University, Stanford, California

Stone, Marshall Harvey, 1938 (1), 303 Eckhart Hall, University of Chicago, Chicago 37, Illinois

Stratton, Julius Adams, 1950 (4), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Street, Jabez Curry, 1953 (3), Jefferson Physical Laboratory, Harvard University, Cambridge 38, Massachusetts

Struve, Otto, 1937 (2), Department of Astronomy, University of California, Berkeley 4, California

Sturtevant, Alfred Henry, 1930 (8), California Institute of Technology, Pasadena 4, California

Suits, Chauncey Guy, 1946 (4), General Electric Research Laboratory, The Knolls, Schenectady, New York

Sumner, James Batcheller, 1948 (9), Savage Hall, New York State College of Agriculture, Cornell University, Ithaca, New York

Taliaferro, William Hay, 1940 (8), University of Chicago, 5724 Ellis Avenue, Chicago 37, Illinois

Tatum, Edward Lawrie, 1952 (9), Department of Biological Sciences, Stanford University, Stanford, California

Teller, Edward, 1948 (3), Department of Physics, University of California, Berkeley 4, California

Terman, Frederick Emmons, 1946 (4), School of Engineering, Stanford University, Stanford, California

Terman, Lewis Madison, 1928 (12), 761 Dolores Street, Stanford University, Stanford, California

Thimann, Kenneth Vivian, 1948 (7), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Thom, Charles, 1937 (7), Port Jefferson, New York

Thomas, Charles Allen, 1948 (4), Monsanto Chemical Company, 1700 South Second Street, St. Louis 4, Missouri

Thomas, Tracy Yerkes, 1941 (1), Swain Hall, Indiana University, Bloomington, Indiana

Thompson, Thomas Gordon, 1951 (13), Department of Oceanography, University of Washington, Seattle 5, Washington

Thurstone, Louis Leon, 1938 (12), Psychometric Laboratory, University of North Carolina, Chapel Hill, North Carolina

Tillett, William Smith, 1951 (10), New York University College of Medicine, 477 First Avenue, New York 16, New York

Timoshenko, Stephen Prokop, 1940 (4), Room 262, Engineering Building, Stanford University, Stanford, California

Tishler, Max, 1953 (5), Process Research and Development Division, Merck & Co., Inc., Rahway, New Jersey

Tolman, Edward Chace, 1937 (12), Department of Psychology, University of California, Berkeley 4, California

Tozzer, Alfred Marston, 1942 (11), Peabody Museum, Harvard University, Cambridge 38, Massachusetts

Trumpler, Robert Julius, 1932 (2), Route 2, Box 1750, Aptos, California

Tuve, Merle Antony, 1946 (3), Department of Terrestrial Magnetism, Carnegie Institution of Washington, 5241 Broad Branch Road, N. W., Washington 15, D. C.

Twitty, Victor Chandler, 1950 (8), Department of Biological Sciences, Stanford University, Stanford, California

Urey, Harold Clayton, 1935 (5), Institute for Nuclear Studies, University of Chicago, Chicago 37, Illinois

Van Niel, Cornelis Bernardus, 1945 (7), Hopkins Marine Station of Stanford University, Pacific Grove, California

Van Slyke, Donald Dexter, 1921 (9), Brookhaven National Laboratory, Upton, Long Island, New York

Van Vleck, John Hasbrouck, 1935 (3), Lyman Laboratory of Physics, Harvard University, Cambridge 38, Massachusetts

Veblen, Oswald, 1919 (1), The Institute for Advanced Study, Princeton, New Jersey

Vestine, Ernest Harry, 1954 (13), Department of Terrestrial Magnetism, Carnegie Institution of Washington, 5241 Broad Branch Road, N. W., Washington 15, D. C.

Vickery, Hubert Bradford, 1943 (9), Connecticut Agricultural Experiment Station, New Haven 4, Connecticut

von Kármán, Theodore, 1938 (4), 1501 South Marengo Avenue, Pasadena, California

von Neumann, John, 1937 (1), The Institute for Advanced Study, Princeton, New Jersey

Waksman, Selman Abraham, 1942 (7), Institute of Microbiology, Rutgers University, New Brunswick, New Jersey

Wald, George, 1950 (9), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Walker, John **Charles,** 1945 (7), 206 Horticulture Building, University of Wisconsin, Madison 6, Wisconsin

Walsh, Joseph Leonard, 1936 (1), Widener 474, Harvard University, Cambridge 38, Massachusetts

Webster, David Locke, 1923 (3), 1830 Cowper Street, Palo Alto., California

Weiss, Paul Alfred, 1947 (8), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Weisskopf, Victor Frederick, 1952 (3), Department of Physics, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Went, Frits Warmolt, 1947 (7), California Institute of Technology, Pasadena 4, California

Werkman, Chester Hamlin, 1946 (9), Department of Bacteriology, Iowa State College, Ames, Iowa

Westheimer, Frank Henry, 1954 (5), Department of Chemistry, Harvard University, 12 Oxford Street, Cambridge 38, Massachusetts

Wetmore, Alexander, 1945 (8), Smithsonian Institution, Washington 25, D. C.

Wetmore, Ralph Hartley, 1954 (7), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Wever, Ernest Glen, 1940 (12), Princeton University, Princeton, New Jersey

Weyl, Claus Hugo Hermann, 1940 (1), The Institute for Advanced Study, Princeton, New Jersey

Wheeler, John Archibald, 1952 (3), Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

Whipple, George Hoyt, 1929 (10), School of Medicine and Dentistry, University of Rochester, 260 Crittenden Boulevard, Rochester 7, New York

Whitehead, John Boswell, 1932 (4), Johns Hopkins University, Baltimore 18, Maryland

Whitford, Albert Edward, 1954 (2), Washburn Observatory, University of Wisconsin, Madison 6, Wisconsin

Whitney, Hassler, 1945 (1), The Institute for Advanced Study, Princeton, New Jersey

Whyburn, Gordon Thomas, 1951 (1), Department of Mathematics, University of Virginia, Charlottesville, Virginia

Wiggers, Carl John, 1951 (9), Cleveland Clinic, 2020 East 93rd Street, Cleveland 6, Ohio

Wigner, Eugene Paul, 1945 (3), 8 Ober Road, Princeton, New Jersey

Williams, Howel, 1950 (6), Department of Geological Sciences, University of California, Berkeley 4, California

Williams, John Warren, 1952 (5), Department of Chemistry, University of Wisconsin, Madison 6, Wisconsin

Williams, Robert R., 1945 (5), 297 Summit Avenue, Summit, New Jersey

Williams, Roger John, 1946 (5), Biochemical Institute, University of Texas, Austin 12, Texas

Willier, Benjamin Harrison, 1945 (8), Department of Biology, Johns Hopkins University, Baltimore 18, Maryland

Wilson, Edgar Bright, Jr., 1947 (5), Department of Chemistry, Harvard University, 12 Oxford Street, Cambridge 38, Massachusetts

Wilson, Edwin Bidwell, 1919 (3), Harvard School of Public Health, 695 Huntington Avenue, Boston 15, Massachusetts

Wilson, Ralph Elmer, 1950 (2), 605 Acacia Avenue, Corona del Mar, California

Wilson, Robert Erastus, 1947 (4), 910 South Michigan Avenue, Chicago 80, Illinois

Wintersteiner, Oskar, 1950 (9), Squibb Institute for Medical Research, New Brunswick, New Jersey

Wislocki, George Bernays, 1941 (8), Harvard Medical School, 25 Shattuck Street, Boston 15, Massachusetts

Wolfrom, Melville Lawrence, 1950 (5), Department of Chemistry, Ohio State University, Columbus 10, Ohio

Wood, Harland Goff, 1953 (9), Department of Biochemistry, School of Medicine, Western Reserve University, Cleveland 6, Ohio

Wood, Robert Williams, 1912 (3), Johns Hopkins University, Baltimore 18, Maryland

Woodring, Wendell Phillips, 1946 (6), United States Geological Survey, Department of the Interior, Washington 25, D. C.

Woodward, Robert Burns, 1953 (5), Department of Chemistry, Harvard University, 12 Oxford Street, Cambridge 38, Massachusetts

Woodworth, Robert Sessions, 1921 (12), Columbia University, New York 27, New York

Woolley, Dilworth Wayne, 1952 (9), Rockefeller Institute for Medical Research, 66th Street and York Avenue, New York 21, New York

Wright, Sewall Green, 1934 (8), Department of Zoology, University of Chicago, Chicago 37, Illinois

Wright, William Hammond, 1922 (2), 60 North Keeble Avenue, San Jose 26, California

Wulf, Oliver Reynolds, 1949 (13), United States Weather Bureau, Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena 4, California

Wyckoff, Ralph Walter Graystone, 1949 (5), Laboratory of Physical Biology, National Institutes of Health, Bethesda 14, Maryland

Yerkes, Robert Mearns, 1923 (12), 4 St. Ronan Terrace, New Haven 11, Connecticut

Yost, Don Merlin Lee, 1944 (5), California Institute of Technology, Pasadena 4, California

Young, William Gould, 1951 (5), Department of Chemistry, University of California, Los Angeles 24, California

Zachariasen, Frederik **William** Houlder, 1949 (3), Department of Physics, University of Chicago, Chicago 37, Illinois

Zariski, Oscar, 1944 (1), Department of Mathematics, Harvard University, Cambridge 38, Massachusetts

Zworykin, Vladimir Kosma, 1943 (4), RCA Laboratories Division, David Sarnoff Research Center, Princeton, New Jersey

Number of Members July 1, 1954: 528.

MEMBERS EMERITI

Anderson, John August, 1928 (2), P. O. Box 332, Pasadena 17, California

Bailey, Liberty Hyde, 1917 (7), Ithaca, New York

Benedict, Francis Gano, 1914 (9), Machiasport, Maine

Child, Charles Manning, 1935 (8), School of Biological Sciences, Jordan Hall, Stanford University, Stanford, California

Hulett, George Augustus, 1922 (5), 44 Washington Road, Princeton, New Jersey

Merrill, Elmer Drew, 1923 (7), Arnold Arboretum, Jamaica Plain, Massachusetts

Novy, Frederick George, 1924 (10), 721 Forest Avenue, Ann Arbor, Michigan

Palache, Charles, 1934 (6), Box 205-D, R. D. 3, Charlottesville, Virginia

Parker, George Howard, 1913 (8), Museum of Comparative Zoology at Harvard College, Oxford Street, Cambridge 38, Massachusetts

Stratton, George Malcolm, 1928 (10), 1023 Life Sciences Building, University of California, Berkeley 4, California

Swanton, John Reed, 1932 (11), 22 George Street, Newton 58, Massachusetts

Tyzzer, Ernest Edward, 1942 (10), 175 Water Street, Wakefield, Massachusetts

Vandiver, Harry Shultz, 1934 (1), Box 7881, University of Texas, Austin 12, Texas

Whitney, Willis Rodney, 1917 (5), General Electric Company, Schenectady 5, New York

Number of Members Emeriti July 1, 1954: 14.

FOREIGN ASSOCIATES

The number in parentheses following the year of election indicates association within the sections of the National Academy of Sciences.

Adrian, Edgar Douglas, 1941 (9), Trinity College, Cambridge, England

Alexandroff, Paul A., 1947 (1), Mathematical Institute of the Academy of Sciences of the U.S.S.R., Bolshaya Kalushskaya 19, Moscow, U.S.S.R.

Bailey, Sir Edward, 1944 (6), 19 Greenhill Gardens, Edinburgh 10, Scotland

Bartlett, Sir Frederic Charles, 1947 (12), University of Cambridge, The Psychological Laboratory, Downing Place, Cambridge, England

Best, Charles Herbert, 1950 (9), Banting and Best Department of Medical Research, University of Toronto, Toronto, Canada

Bjerrum, Niels, 1952 (5), The Royal Veterinary and Agricultural College, Copenhagen, Denmark

Bohr, Niels, 1925 (3), Institute for Theoretical Physics, Blegdamsvej 15, Copenhagen, Denmark

Bordet, Jules, 1935 (9), Pasteur Institute, Rue du Remorquer, 28, Brussels, Belgium

Bragg, Sir William Lawrence, 1945 (3), Cavendish Laboratory, The University, Cambridge, England

de Broglie, Prince Louis, 1948 (3), 94 Rue Perronet, Neuilly-sur-Seine, France

Burnet, Sir Macfarlane, 1954 (10), The Walter and Eliza Hall Institute of Medical Research, Melbourne, Australia

Caso, Alfonso, 1943 (11), Av. Sudderman No. 312, Col. Chapultepec-Morales, Mexico 5, D. F.

Chapman, Sydney, 1946 (13), The Geophysical Institute, College, Alaska, U.S.A.

Dale, Sir Henry Hallett, 1940 (9), The Wellcome Trust, 28 Portman Square, London, W.1., England

Dirac, Paul Adrien Maurice, 1949 (1), Department of Mathematics, St. John's College, Cambridge, England

Eskola, Pentti Eelis, 1951 (6), Helsinki University, Snellmanink. 5, Helsinki, Finland

Fisher, Sir Ronald Aylmer, 1948 (8), Department of Genetics, University of Cambridge, 44 Storey's Way, Cambridge, England

von Frisch, Karl, 1951 (8), The Zoological Institute, University of Munich, Munich, Germany

Hadamard, Jacques, 1926 (1), 12 Rue Emile Faguet, Paris XIV, France

Helland-Hansen, Björn, 1947 (13), Chr. Michelsens Institutt for Videnscap, Bergen, Norway

Hill, Archibald Vivian, 1941 (9), 16 Bishopswood Road, Highgate, London, N.6, England

Hill, James Peter, 1940 (8), Kanimbla, Dollis Avenue, London, N.3, England

Houssay, Bernardo Alberto, 1940 (9), Viamonte 2790, Buenos Aires, Argentina

Jeffreys, Harold, 1945 (13), St. John's College, Cambridge, England

Jones, Sir Harold Spencer, 1943 (2), Royal Greenwich Observatory, Herstmonceux Castle, Hailsham, Sussex, England

Kapitza, Peter Leonidovich, 1946 (3), Institute for Physical Problems, Academy of Sciences of the U.S.S.R., Moscow, U.S.S.R.

Karrer, Paul, 1945 (5), University of Zurich, Zurich, Switzerland

Keith, Sir Arthur, 1941 (11), Buckston Browne Farm, Downe, Farnborough, Kent, England

Kluyver, Albert Jan, 1950 (7), Technical University, Delft, The Netherlands

Levi, Giuseppe, 1940 (8), Instituto di Anatomia Umana, Corso Massimo D'Azeglio, 52, Turin, Italy

Lim, Robert K. S., 1942 (9), The Miles-Ames Research Laboratory, Elkhart, Indiana

Linderstrøm-Lang, Kaj Ulrik, 1947 (5), Chemical Division, Carlsberg Laboratory, Copenhagen, Denmark

Oort, Jan Hendrik, 1953 (1), Observatory of Leiden, Leiden, The Netherlands

Penfield, Wilder, 1953 (10), Montreal Neurological Institute, 3801 University Street, Montreal 2, Canada

Pieron, Henri, 1949 (12), Institute of Psychology, University of Paris, Paris, France

Portevin, Albert M. G. R., 1954 (4), 21, Boulevard de Beauséjour, Paris 16, France

Reichstein, Tadeus, 1952 (5), Organisch-chemische Anstalt, St. Johanns-Ring 19, Basel, Switzerland

Renner, Otto, 1954 (7), The Botanical Institute, University of Munich, Munich, Germany

Robinson, Sir Robert, 1934 (5), Dyson Perrins Laboratory, South Parks Road, Oxford, England

Ruzicka, Leopold, 1944 (5), Department of Organic Chemistry, Institute of Technology, Zurich, Switzerland

Southwell, Sir Richard Vynne, 1943 (4), The Old House, Trumpington, Cambridge, England

Svedberg, The, 1945 (5), Fysikalisk-Kemiska Institutionen, Uppsala University, Uppsala, Sweden

Sverdrup, Harald Ulrik, (1945)* 1952 (13), Norsk Polarinstitutt, Observatoriegt 1, Oslo, Norway

Taylor, Sir Geoffrey I., 1945 (1), Trinity College, Cambridge, England

Thomson, Sir Godfrey, 1951 (12), Edinburgh University, 5 Ravelston Dykes, Edinburgh 4, Scotland

* Dr. Sverdrup was elected a member of the Academy in 1945. He resigned on April 2, 1951, on which date he had maintained residence in Norway for three years, thereby losing his American citizenship. He was elected a foreign associate on April 29, 1952.

Tiselius, Arne W. K., 1949 (9), Institute of Biochemistry, Uppsala University, Uppsala, Sweden

Vallee-Poussin, C. de la, 1929 (1), University of Louvain, Louvain, Belgium

Vening Meinesz, Felix Andries, 1939 (13), Potgieterlaan 5, Amersfoort, The Netherlands

Watson, D. M. S., 1938 (8), University College, Gower Street, London, W.C.1, England

Wieland, Heinrich, 1932 (5), Sophienstrasse 9, Munich 2 NW, Germany

Winge, Öjvind, 1949 (9), Department of Physiology, Carlsberg Laboratory, Copenhagen (Valby), Denmark

Yukawa, Hideki, 1949 (3), Yukawa Hall, Kyoto University, Kyoto, Japan

Number of Foreign Associates July 1, 1954: 52

SECTIONS

(1) *Mathematics*—29 members

McShane, E. J., *Chairman* (1955)
Ahlfors, L. V.
Albert, A. A.
Alexander, J. W.
Bell, E. T.
Bochner, S.
Chevalley, C.
Coble, A. B.
Douglas, Jesse
Eisenhart, L. P.
Evans, G. C.
Hille, Einar
Jacobson, Nathan
Kasner, Edward
Lefschetz, Solomon
Mac Lane, Saunders
Moore, R. L.
Morse, Marston
Murnaghan, F. D.
Smith, Paul A.
Stone, M. H.
Thomas, T. Y.
Veblen, Oswald
von Neumann, John
Walsh, J. L.
Weyl, Hermann
Whitney, Hassler
Whyburn, G. T.
Zariski, O.

Foreign Associates

Alexandroff, P. A.
Dirac, P. A. M.
Hadamard, Jacques
Taylor, Sir Geoffrey
Valle-Poussin, C. de la

(2) *Astronomy*—27 members

Struve, Otto, *Chairman* (1956)
Abbot, C. G.
Adams, W. S.
Babcock, H. D.
Babcock, H. W.
Bowen, I. S.
Brouwer, Dirk
Clemence, G. M.
Joy, A. H.
King, A. S.
Kuiper, G. P.
Mayall, N. U.
Menzel, D. H.
Merrill, P. W.
Mitchell, S. A.
Nicholson, S. B.
Ross, F. E.
Russell, H. N.
Seares, F. H.
Shapley, Harlow
Slipher, V. M.
Spitzer, Lyman, Jr.
Stebbins, Joel
Trumpler, R. J.
Whitford, A. E.
Wilson, Ralph E.
Wright, W. H.

Foreign Associates

Jones, Sir H. Spencer
Oort, Jan Hendrik

(3) *Physics*—77 members

Van Vleck, J. H., *Chairman* (1957)
Allison, S. K.
Alvarez, L. W.
Anderson, C. D.
Bacher, R. F.
Bainbridge, K. T.
Bardeen, John
Beams, J. W.
Bethe, H. A.
Birge, R. T.
Bloch, Felix
Bradbury, N. E.
Breit, Gregory
Bridgman, P. W.
Brillouin, Leon
Brode, R. B.
Coblentz, W. W.
Compton, A. H.
Condon, E. U.
Coolidge, W. D.
Davis, Bergen
Davisson, C. J.
Dennison, D. M.
DuBridge, L. A.
DuMond, J. W. M.
Dunning, J. R.
Einstein, Albert
Epstein, P. S.
Fermi, Enrico
Feynman, R. P.
Franck, James
Gamow, George
Goudsmit, S. A.
Houston, W. V.
Hull, A. W.
Kemble, E. C.
Kent, R. H.
Kerst, D. W.
Lamb, W. E., Jr.
Land, E. H.
Lauritsen, C. C.
Lawrence, E. O.
Loomis, F. W.
Lyman, Theodore
Mason, Max
McMillan, E. M.
Meggers, W. F.
Mulliken, R. S.
Nier, A. O. C.
O'Brien, Brian
Oppenheimer, J. R.
Panofsky, W. K. H.
Pegram, G. B.

Pierce, G. W.
Purcell, E. M.
Rabi, I. I.
Ramsey, N. F.
Robertson, H. P.
Rossi, Bruno
Saunders, F. A.
Schwinger, Julian
Segrè, E.
Seitz, Frederick
Serber, R.
Slater, J. C.
Stern, Otto
Stewart, G. W.
Street, J. C.
Teller, Edward
Tuve, M. A.
Webster, D. L.
Weisskopf, V. F.
Wheeler, J. A.
Wigner, E. P.
Wilson, Edwin B.
Wood, R. W.
Zachariasen, W. H.

Foreign Associates

Bohr, Niels
Bragg, Sir Lawrence
de Broglie, Prince Louis
Kapitza, P. L.
Yukawa, Hideki

(4) *Engineering*—39 members

Terman, F. E., *Chairman* (1956)
Adams, C. A.
Bain, E. C.
Briggs, L. J.
Buckley, O. E.
Bush, Vannevar
Cochrane, E. L.
Curme, G. O., Jr.
Den Hartog, J. P.
Dryden, H. L.
Durand, W. F.
Fisk, J. B.
Fletcher, Harvey
Foote, P. D.
Gibbs, W. F.
Gilliland, E. R.
Greenewalt, C. H.
Hoover, Herbert
Hunsaker, J. C.
Jeffries, Zay
Kelly, M. J.
Kettering, C. F.
Lewis, W. K.
Loomis, A. L.
Merica, P. D.
Murphree, E. V.
Raymond, A. E.
Savage, J. L.
Shockley, W.
Slepian, Joseph
Soderberg, C. R.
Stratton, J. A.
Suits, C. G.
Thomas, C. A.
Timoshenko, Stephen
von Kármán, T.
Whitehead, J. B.
Wilson, Robert E.
Zworykin, V. K.

Foreign Associate

Southwell, Sir Richard V.

(5) *Chemistry*—72 members

Kistiakowsky, G. B., *Chairman* (1956)
Adams, Roger
Badger, R. M.
Bartlett, P. D.
Bolton, E. K.
Brode, W. R.
Calvin, Melvin
Conant, J. B.
Cope, A. C.
Craig, L. C.
Daniels, Farrington
Debye, Peter
Edsall, J. T.
Elderfield, R. C.
Eyring, Henry
Fieser, L. F.
Flory, P. J.
Folkers, Karl
Fuoss, R. M.
Fuson, R. C.
Giauque, W. F.
Gilman, Henry
Hammett, L. P.
Harned, H. S.
Hendricks, S. B.
Hildebrand, J. H.
Hirschfelder, J. A.
Jacobs, W. A.
Johnson, J. R.
Johnson, W. S.
Keyes, F. G.
Kharasch, M. S.
Kimball, G. E.
Kirkwood, J. G.
Kraus, C. A.
La Mer, V. K.
Langmuir, Irving
Latimer, W. M.
Libby, W. F.
Lind, S. C.
Longsworth, L. G.
MacInnes, D. A.
Marvel, C. S.
Mayer, J. E.
McElvain, S. M.
Mees, C. E. K.
Niemann, C. G.

Noyes, W. A., Jr.
Onsager, Lars
Pauling, Linus
Pitzer, K. S.
Rodebush, W. H.
Rossini, F. D.
Scatchard, George
Schlesinger, H. I.
Seaborg, G. T.
Shedlovsky, Theodore
Small, L. F.
Smith, L. I.
Spedding, F. H.
Tishler, Max
Urey, H. C.
Westheimer, F. H.
Williams, J. W.
Williams, Robert R.
Williams, Roger J.
Wilson, E. Bright, Jr.
Wolfrom, M. L.
Woodward, R. B.
Wyckoff, R. W.
Yost, D. M.
Young, W. G.

Foreign Associates

Bjerrum, Niels
Karrer, Paul
Linderstrøm-Lang, K. U.
Reichstein, Tadeus
Robinson, Sir Robert
Ruzicka, Leopold
Svedberg, The
Wieland, Heinrich

(6) *Geology*—36 members

Gilluly, James, *Chairman* (1957)
Allen, E. T.
Berkey, C. P.
Birch, Francis
Blackwelder, Eliot
Bowen, N. L.
Bradley, W. H.
Bramlette, M. N.
Bucher, W. H.
Buddington, A. F.
Buerger, M. J.
Chaney, R. W.
Cloos, Ernst
Daly, R. A.
Day, A. L.
DeGolyer, E. L.
Dunbar, C. O.
Gregory, W. K.
Hess, H. H.
Hewett, D. F.
Kelley, W. P.
Knopf, Adolph
Larsen, E. S., Jr.
Leith, C. K.
Longwell, C. R.
Lovering, T. S.
Mead, W. J.
Mendenhall, W. C.
Nolan, T. B.
Reeside, J. B., Jr.
Rubey, W. W.
Ruedemann, Rudolf
Schairer, J. F.
Simpson, G. G.
Williams, Howel
Woodring, W. P.

Foreign Associates

Bailey, Sir Edward
Escola, Pentti

(7) *Botany*—42 members

Raper, K. B., *Chairman* (1956)
Anderson, Edgar
Babcock, E. B.
Bailey, I. W.
Barker, H. A.
Beadle, G. W.
Blakeslee, A. F.
Bonner, James
Brink, R. A.
Burkholder, P. R.
Chandler, W. H.
Clausen, R. E.
Cleland, R. E.
Couch, J. N.
Delbrück, Max
Dodge, B. O.
Duggar, B. M.
Emerson, Robert
Fred, E. B.
Goddard, D. R.
Horsfall, J. G.
Jones, D. F.
Kaufmann, B. P.
Kunkel, L. O.
Mangelsdorf, P. C.
McClintock, Barbara
Osterhout, W. J. V.
Rhoades, M. M.
Riker, A. J.
Robbins, W. J.
Sax, Karl
Sinnott, E. W.
Smith, Gilbert M.
Stakman, E. C.
Stebbins, G. L., Jr.
Thimann, K. V.
Thom, Charles
Van Niel, C. B.
Waksman, S. A.
Walker, J. C.
Went, F. W.
Wetmore, R. H.

Foreign Associate

Kluyver, A. J.

(8) *Zoölogy and Anatomy*—50 members

Willier, B. H., *Chairman* (1955)
Allee, W. C.
Bartelmez, G. W.
Bigelow, H. B.
Bloom, William
Bodine, J. H.
Castle, W. E.
Cleveland, L. R.
Corner, G. W.
Danforth, C. H.
Demerec, Milislav
Detwiler, S. R.
Dobzhansky, Th.
Dunn, L. C.
Goldschmidt, R. B.
Hamburger, Viktor
Harrison, R. G.
Hartman, C. G.
Harvey, E. N.
Herrick, C. J.
Hisaw, F. L.
Hubbs, C. L.
Hutchison, G. E.
Irwin, M. R.
Jacobs, M. H.
Kellogg, Remington
Lewis, W. H.
Mayr, Ernst
Metz, C. W.
Mirsky, A. E.
Moore, C. R.
Muller, H. J.
Nicholas, J. S.
Painter, T. S.
Patterson, J. T.
Petrunkevitch, Alexander
Riddle, Oscar
Romer, A. S.
Schmitt, F. O.
Schrader, Franz
Smith, Philip E.
Sonneborn, T. M.
Stern, Curt
Sturtevant, A. H.
Taliaferro, W. H.
Twitty, V. C.
Weiss, Paul
Wetmore, Alexander
Wislocki, G. B.
Wright, Sewall

Foreign Associates

Fisher, R. A.
von Frisch, Karl
Hill, J. P.
Levi, Giuseppe
Watson, D. M. S.

(9) *Physiology and Biochemistry*—62 members

Fenn, W. O., *Chairman* (1957)
Anderson, R. J.
Ball, E. G.
Balls, A. K.
Bard, Philip
Bronk, D. W.
Carlson, A. J.
Carter, H. E.
Chance, Britton
Clark, W. M.
Clarke, H. T.
Cori, Carl F.
Cori, Gerty T.
Davis, Hallowell
Doisy, E. A.
DuBois, E. F.
du Vigneaud, Vincent
Elvehjem, C. A.
Erlanger, Joseph
Evans, H. M.
Fischer, H. O. L.
Forbes, Alexander
Fruton, J. S.
Gasser, H. S.
Hartline, H. K.
Hastings, A. B.
Kendall, E. C.
King, C. G.
Landis, E. M.
Link, K. P.
Lipmann, Fritz
Lloyd, D. P. C.
Loeb, R. F.
Long, C. N. H.
Lorente de Nó, R.
Mann, F. C.
Marshall, E. K., Jr.
Maynard, L. A.
McCollum, E. V.
Meek, W. J.
Northrop, J. H.
Oncley, J. L.
Peters, J. P.
Richards, A. N.
Rittenberg, David
Rose, W. C.
Schmidt, C. F.
Shaffer, P. A.
Sherman, H. C.
Smith, Homer W.
Stadie, W. C.
Stanley, W. M.
Sumner, J. B.
Tatum, E. L.
Van Slyke, D. D.
Vickery, H. B.
Wald, George
Werkman, C. H.
Wiggers, C. J.
Wintersteiner, Oskar
Wood, H. G.
Woolley, D. W.

Foreign Associates

Adrian, E. D.
Best, C. H.
Bordet, Jules
Dale, Sir Henry
Hill, A. V.
Houssay, B. A.
Lim, Robert K. S.
Tiselius, Arne W. K.
Winge, Öjvind

(10) *Pathology and Bacteriology*—36 members

Cannon, P. R., *Chairman* (1957)
Albright, Fuller
Armstrong, Charles
Avery, O. T.
Bailey, P.
Blalock, Alfred
Castle, W. B.
Coggeshall, L. T.
Cole, Rufus
Dochez, A. R.
Dragstedt, L. R.
Dubos, R. J.
Enders, J. F.
Francis, Thomas, Jr.
Gamble, J. L.
Goodpasture, E. W.
Graham, E. A.
Heidelberger, Michael
Horsfall, F. L., Jr.
Huggins, C. B.
Little, C. C.
Loeb, Leo
Long, E. R.
Maxcy, K. F.
McMaster, P. D.
Meyer, K. F.
Opie, E. L.
Paul, J. R.
Rich, A. R.
Rivers, T. M.
Robertson, O. H.
Rous, Peyton
Sabin, Albert B.
Shope, R. E.
Tillett, W. S.
Whipple, G. H.

Foreign Associate

Penfield, Wilder

(11) *Anthropology*—10 members

Shapiro, H. L., *Chairman* (1957)
Kidder, A. V.
Kluckhohn, C. K. M.
Kroeber, A. L.
Lothrop, S. K.
Lowie, R. H.
Schultz, A. H.
Spier, Leslie
Steward, J. H.
Tozzer, A. M.

Foreign Associates

Caso, Alfonso
Keith, Sir Arthur

(12) *Psychology*—25 members

Graham, C. H., *Chairman* (1956)
Beach, F. A.
Boring, E. G.
Carmichael, Leonard
Gesell, Arnold
Guilford, J. P.
Harlow, H. F.
Hilgard, E. R.
Hunter, W. S.
Köhler, Wolfgang
Lashley, K. S.
Lindsley, D. B.
Miles, W. R.
Nissen, H. W.
Pillsbury, W. B.
Richter, C. P.
Skinner, B. F.
Stevens, S. S.
Stone, C. P.
Terman, L. M.
Thurstone, L. L.
Tolman, E. C.
Wever, E. G.
Woodworth, R. S.
Yerkes, R. M.

Foreign Associates

Bartlett, Sir Frederic
Pieron, Henri
Thomson, Sir Godfrey

(13) *Geophysics*—23 members

Berkner, L. V., *Chairman* (1957)
Adams, L. H.
Benioff, Hugo
Bjerknes, J.
Byerly, Perry
Byers, H. R.
Eckart, Carl
Ewing, Maurice
Fleming, J. A.
Griggs, D. T.
Gunn, Ross
Gutenberg, Beno
Iselin, C. O'D.
Lambert, W. D.
Macelwane, J. B.
Pekeris, C. L.
Piggot, C. S.
Reichelderfer, F. W.
Rossby, C.-G.
Slichter, L. B.
Thompson, T. G.
Vestine, E. H.
Wulf, O. R.

Foreign Associates

Chapman, Sydney
Helland-Hansen, Björn
Jeffreys, H.
Sverdrup, H. U.
Vening Meinesz, F. A.

GEOGRAPHICAL LISTING of ACADEMY MEMBERS

Arizona

Slipher, V. M.

California

Adams, W. S.
Alvarez, L. W.
Anderson, C. D.
Anderson, J. A.
Babcock, E. B.
Babcock, H. D.
Babcock, H. W.
Bacher, R. F.
Badger, R. M.
Barker, H. A.
Beadle, G. W.
Bell, E. T.
Benioff, Hugo
Birge, R. T.
Bjerknes, J.
Blackwelder, Eliot
Bloch, Felix
Bonner, James
Bowen, I. S.
Bramlette, M. N.
Brode, R. B.
Byerly, Perry
Calvin, Melvin
Castle, W. E.
Chandler, W. H.
Chaney, R. W.
Child, C. M.
Clausen, R. E.
Coble, A. B.
Danforth, C. H.
Delbrück, Max
DuBridge, L. A.
DuMond, J. W. M.
Eckart, Carl
Epstein, P. S.
Evans, G. C.
Evans, H. M.
Feynman, R. P.
Fischer, H. O. L.
Giauque, W. F.
Goldschmidt, R. B.
Griggs, D. T.
Guilford, J. P.
Gutenberg, Beno
Hewett, D. F.
Hildebrand, J. H.
Hilgard, E. R.
Hubbs, C. L.
Joy, A. H.
Kelley, W. P.
King, A. S.
Knopf, Adolph
Kroeber, A. L.
Lamb, W. E., Jr.
Latimer, W. M.
Lauritsen, C. C.
Lawrence, E. O.
Lindsley, D. B.
Lowie, R. H.
Mason, Max
Mayall, N. U.
McMillan, E. M.
Merrill, P. W.
Meyer, K. F.
Nicholson, S. B.
Niemann, Carl
Northrop, J. H.
Panofsky, W. K. H.
Pauling, Linus
Pitzer, K. S.
Raymond, A. E.
Robertson, O. H.
Ross, F. E.
Seaborg, G. T.
Seares, F. H.
Segrè, Emilio
Slichter, L. B.
Smith, G. M.
Spier, Leslie
Stanley, W. M.
Stebbins, G. L., Jr.
Stebbins, Joel
Stern, Curt
Stern, Otto
Stone, C. P.
Stratton, G. M.
Struve, Otto
Sturtevant, A. H.
Tatum, E. L.
Teller, Edward
Terman, F. E.
Terman, L. M.
Timoshenko, S. P.
Tolman, E. C.
Trumpler, R. J.
Twitty, V. C.
Van Niel, C. B.
von Kármán, T.

Webster, D. L.
Went, F. W.
Williams, Howel
Wilson, Ralph E.
Wright, W. H.
Wulf, O. R.
Yost, D. M. L.
Young, W. G.

Colorado

Gilluly, James
Lovering, T. S.
Savage, J. L.

Connecticut

Anderson, R. J.
Beach, F. A.
Breit, G.
Brouwer, Dirk
Dunbar, C. O.
Fruton, J. S.
Fuoss, R. M.
Gesell, Arnold
Harned, H. S.
Harrison, R. G.
Hille, Einar
Horsfall, J. G.
Hutchinson, G. E.
Jacobson, Nathan
Jones, D. F.
Kirkwood, J. G.
Lambert, W. D.
Long, C. N. H.
Longwell, R. C.
Miles, W. R.
Nicholas, J. S.
Onsager, Lars
Paul, J. R.
Peters, J. P.
Petrunkevitch, A.
Sinnott, E. W.
Vickery, H. B.
Yerkes, R. M.

Delaware

Bolton, E. K.
Greenewalt, C. H.

District of Columbia

Abbot, C. G.
Adams, L. H.
Bowen, Norman L.
Bradley, W. H.
Briggs, L. J.
Brode, W. R.
Bush, V.
Carmichael, Leonard
Clemence, G. M.
Coblentz, W. W.
Dryden, H. L.
Gamow, George
Gunn, Ross
Kellogg, Remington
Larsen, E. S., Jr.
Leith, C. K.
Meggers, W. F.
Nolan, T. B.
Piggot, C. S.
Reichelderfer, F. W.
Rubey, W. W.
Schairer, J. F.
Tuve, M. A.
Vestine, E. H.
Wetmore, Alexander
Woodring, W. P.

Florida

Allee, W. C.
Lashley, K. S.
Nissen, H. W.
Riddle, Oscar

Georgia

Burkholder, P. R.

Illinois

Adams, Roger
Albert, A. A.
Allison, S. K.
Bailey, P.
Bardeen, John
Bloom, William
Byers, H. R.
Cannon, Paul
Carlson, A. J.
Carter, H. E.
Coggeshall, L. T.
Dragstedt, L. R.
Emerson, Robert
Fermi, Enrico
Franck, James
Fuson, R. C.
Huggins, Charles
Kerst, D. W.
Kharasch, M. S.
Libby, W. F.
Loomis, F. W.
Mac Lane, Saunders
Marvel, C. S.
Mayer, J. E.
Moore, C. R.
Mulliken, R. S.
Rhoades, M. M.
Rodebush, W. H.
Rose, W. C.
Schlesinger, H. I.
Seitz, Frederick
Steward, J. H.
Stone, M. H.
Taliaferro, W. H.
Urey, H. C.
Wilson, Robert E.
Wright, Sewall
Zachariasen, W. H.

Indiana

Balls, A. K.
Cleland, R. E.
*Lim, R. K. S.
Muller, H. J.
Sonneborn, T. M.
Thomas, T. Y.

Iowa

Bodine, J. H.
Gilman, Henry
Spedding, F. H.
Stewart, G. W.
Werkman, C. H.

Maine

Benedict, F. G.
Little, C. C.

Maryland

Armstrong, Charles
Bard, Philip
Bartelmez, G. W.
Blalock, Alfred
Clark, W. M.
Cloos, Ernst
Corner, G. W.
Day, A. L.
Hendricks, S. B.

* Foreign Associate.

Kent, R. H.
Marshall, E. K., Jr.
Maxcy, K. F.
McCollum, E. V.
Mendenhall, W. C.
Murnaghan, F. D.
Reeside, J. B., Jr.
Rich, A. R.
Richter, C. P.
Small, L. F.
Whitehead, J. B.
Willier, B. H.
Wood, R. W.
Wyckoff, R. W. G.

Massachusetts

Ahlfors, L. V.
Albright, Fuller
Allen, E. T.
Bailey, I. W.
Bainbridge, K. T.
Ball, E. G.
Bartlett, P. D.
Bigelow, H. B.
Birch, Francis
Blakeslee, A. F.
Boring, E. G.
Bridgman, P. W.
Buerger, M. J.
Castle, W. B.
Cleveland, L. R.
Cochrane, E. L.
Cope, A. C.
Daly, R. A.
Den Hartog, J. P.
Edsall, J. T.
Enders, J. F.
Fieser, L. F.
Forbes, Alexander
Gamble, J. L.
Gilliland, E. R.
Hastings, A. B.
Hisaw, F. L.
Hunsaker, J. C.
Iselin, C. O'D.
Jeffries, Zay
Kemble, E. C.
Keyes, F. G.
Kidder, A. V.
Kistiakowsky, G. B.
Kluckhohn, Clyde
Land, E. H.
Landis, E. M.
Lewis, W. K.
Lipmann, Fritz
Lothrop, S. K.
Lyman, Theodore
Mangelsdorf, P. C.
Mayr, Ernst
Mead, W. J.
Menzel, D. H.
Merrill, E. D.
Oncley, J. L.
Parker, G. H.
Pierce, G. W.
Purcell, E. M.
Ramsey, Norman
Romer, A. S.
Rossi, B. B.
Saunders, F. A.
Sax, Karl
Scatchard, George
Schmitt, F. O.
Schwinger, Julian
Shapley, Harlow
Skinner, B. F.
Slater, J. C.
Soderberg, C. R.
Stevens, S. S.
Stratton, J. A.
Street, J. C.
Swanton, J. R.
Thimann, K. V.
Tozzer, A. M.
Tyzzer, E. E.
Van Vleck, J. H.
Wald, George
Walsh, J. L.
Weisskopf, V. F.
Westheimer, F. H.
Wetmore, R. H.
Wilson, E. Bright, Jr.
Wilson, Edwin B.
Wislocki, G. B.
Woodward, R. B.
Zariski, Oscar

Michigan

Dennison, D. M.
Elderfield, R. C.
Francis, Thomas, Jr.
Herrick, C. J.
Kettering, C. F.
Novy, F. G.
Pillsbury, W. B.

Minnesota

Mann, F. C.
Nier, A. O. C.
Smith, L. I.
Stakman, E. C.

Missouri

Anderson, Edgar
Compton, A. H.
Cori, C. F.
Cori, G. T.
Davis, Hallowell
Doisy, E. A.
Erlanger, Joseph
Graham, E. A.
Hamburger, Viktor
Loeb, Leo
Macelwane, J. B.
Shaffer, P. A.
Thomas, C. A.

New Jersey

Alexander, J. W.
Bochner, S.
Buckley, O. E.
Buddington, A. F.
Einstein, Albert
Eisenhart, L. P.
Fisk, J. B.
Folkers, Karl
Hartman, C. G.
Harvey, E. N.
Hess, H. H.
Hulett, G. A.
Kendall, E. C.
Lefschetz, Solomon
Morse, Marston
Oppenheimer, J. R.
Russell, H. N.
Shockley, W.
Spitzer, Lyman, Jr.
Tishler, Max
Veblen, Oswald
von Neumann, John
Waksman, S. A.
Wever, E. G.
Weyl, Hermann
Wheeler, J. A.

Whitney, Hassler
Wigner, E. P.
Williams, R. R.
Wintersteiner, Oskar
Zworykin, V. K.

New Mexico

Bradbury, N. E.

New York

Bailey, L. H.
Berkey, C. P.
Berkner, L. V.
Bethe, H. A.
Brillouin, Leon
Bronk, D. W.
Bucher, W. H.
Chevalley, C.
Clarke, H. T.
Cole, Rufus
Condon, E. U.
Coolidge, W. D.
Craig, L. C.
Curme, G. O., Jr.
Davis, Bergen
Debye, Peter
Demerec, Milislav
Detwiler, S. R.
Dobzhansky, Th.
Dochez, A. R.
Dodge, B. O.
Douglas, Jesse
DuBois, E. F.
Dubos, R. J.
Duggar, B. M.
Dunn, L. C.
Dunning, J. R.
Durand, W. F.
du Vigneaud, Vincent
Ewing, Maurice
Fenn, W. O.
Flory, P. J.
Gasser, H. S.
Gibbs, W. F.
Goudsmit, S. A.
Graham, C. H.
Gregory, W. K.
Hammett, L. P.
Hartline, H. K.
Heidelberger, Michael
Hoover, Herbert
Horsfall, F. L., Jr.
Hull, A. W.
Jacobs, W. A.
Johnson, J. R.
Kasner, Edward
Kaufmann, B. P.
Kelly, M. J.
Kimball, G. E.
King, C. G.
Kunkel, L. O.
LaMer, V. K.
Langmuir, Irving
Lloyd. D. P. C.
Loeb, R. F.
Longsworth, L. G.
Loomis, A. L.
Lorente de Nó, R.
MacInnes, D. A.
Maynard, L. A.
McClintock, Barbara
McMaster, P. D.
Mees, C. E. K.
Merica, P. D.
Mirsky, A. E.
Murphree, E. V.
Noyes, W. A., Jr.
Opie, E. L.
Osterhout, W. J. V.
Pegram, G. B.
Rabi, I. I.
Rittenberg, David
Rivers, T. M.
Robbins, W. J.
Rous, Peyton
Ruedemann, Rudolf
Schrader, Franz
Serber, Robert
Shapiro, H. L.
Shedlovsky, Theodore
Sherman, H. C.
Shope, R. E.
Simpson, G. G.
Smith, H. W.
Smith, P. A.
Smith, P. E.
Suits, C. G.
Sumner, J. B.
Thom, Charles
Tillett, W. S.
Van Slyke, D. D.
Weiss, Paul
Whipple, G. H.
Whitney, W. R.
Woodworth, R. S.
Woolley, D. W.

North Carolina

Couch, J. N.
Thurstone, L. L.

Ohio

Sabin, A. B.
Wiggers, C. J.
Wolfrom, M. L.
Wood, H. G.

Pennsylvania

Adams, C. A.
Bain, E. C.
Chance, Britton
Foote, P. D.
Goddard, D. R.
Jacobs, M. H.
Köhler, Wolfgang
Lewis, W. H.
Long, E. R.
Metz, C. W.
Richards, A. N.
Rossini, F. D.
Schmidt, C. F.
Slepian, Joseph
Stadie, W. C.

Rhode Island

Hunter, W. S.
Kraus, C. A.

Tennessee

Avery, O. T.
Fleming, J. A.
Goodpasture, E. W.
Lind, S. C.

Texas

DeGolyer, E.
Houston, W. V.
Moore, R. L.
Painter, T. S.
Patterson, J. T.
Vandiver, H. S.
Williams, R. J.

Utah

Eyring, Henry
Fletcher, Harvey

Virginia

Beams, J. W.
Davisson, C. J.
McShane, E. J.
Mitchell, S. A.
Palache, Charles
Whyburn, G. T.

Washington

Thompson, T. G.

Wisconsin

Brink, R. A.
Daniels, Farrington
Elvehjem, C. A.
Fred, E. B.
Harlow, H. F.
Hirschfelder, J. O.
Irwin, M. R.
Johnson, W. S.
Kuiper, G. P.
Link, K. P.
McElvain, S. M.
Meek, W. J.
Raper, K. B.
Riker, A. J.
Walker, J. C.
Whitford, A. E.
Williams, J. W.

Members Abroad

France—

Robertson, H. P.

Germany—

Conant, J. B.

Israel—

Pekeris, C. L.

Sweden—

Rossby, C.-G.

Switzerland—

Schultz, A. H.

UNIVERSITY OF MICHIGAN LIBRARIES

INFORMATION TO CONTRIBUTORS

THE PROCEEDINGS is the official organ of the NATIONAL ACADEMY OF SCIENCES and of the NATIONAL RESEARCH COUNCIL for the publication of brief accounts of important current researches of members of the ACADEMY and of the COUNCIL and of other American investigators. The PROCEEDINGS will aim especially to secure prompt publication of original announcements of discoveries and wide circulation of the results of American research among investigators in other countries and in all branches of science.

ARTICLES should be brief. The viewpoint should be comprehensive in giving the relation of the paper to previous publications of the author or of others and in exhibiting, where practicable, the significance of the work for other branches of science. Elaborate technical details of the work and long tables of data should be avoided, but authors should be precise in making clear the new results and should give some record of the methods and data upon which they are based.

MANUSCRIPTS should be prepared with a current number of the PROCEEDINGS as a model in matters of form, and should be typewritten in duplicate with double spacing, the author retaining one copy. Illustrations should be confined to text figures of simple character, though more elaborate illustration may be allowed in special instances to authors willing to pay for their preparation and insertion. Particular attention should be given to arranging tabular matter in a simple and concise manner.

MANUSCRIPTS will be accepted only from members of the Academy or from chairmen of the divisions of the National Research Council who will assume responsibility for the propriety and scientific standards of the paper and should be addressed to *Proceedings of the* NATIONAL ACADEMY OF SCIENCES, Attention Miss Mary D. Alexander, University of Chicago Press, 5750 Ellis Avenue, Chicago 37, Illinois. Articles of which a member of the Academy is author or co-author may extend to 10 printed pages, those by non-members to 5 printed pages; space above these limits will be billed to the author at $21.00 per page. Accompanying the manuscript should be a statement by the author or his representative that he assumes responsibility for the charges if incurred.

GALLEY PROOF will be sent, with the understanding that charges for author's corrections shall be billed to him. Author should therefore make final revisions on the typewritten manuscripts. Page proofs will not be sent.

REPRINTS should be ordered when proof is returned, on the reprint order form which will be inclosed with the proof. The order blank gives a tabular estimation of the cost of reprints; a copy will be supplied at any time upon request.

CONTENTS

Page

ASTRONOMY.—FLUCTUATIONS IN THE SPACE DISTRIBUTION OF THE GALAXIES. . . . . *By Vera Cooper Rubin* 541

ASTRONOMY.—MAGELLANIC CLOUDS. XIII. COMPARISON OF MAGELLANIC AND GALACTIC ECLIPSING VARIABLES . . . . . *By Henry Norris Russell* 549

CHEMISTRY.—THE DIFFUSION COEFFICIENTS OF THE ALKALI METAL CHLORIDES AND POTASSIUM AND SILVER NITRATES IN DILUTE AQUEOUS SOLUTIONS AT 25° . . . . . *By Herbert S. Harned* 551

ENGINEERING.—THE SHOCK TUBE IN AERODYNAMIC AND STRUCTURAL RESEARCH . . . . . *By H. G. Stever and R. L. Bisplinghoff* 557

ENGINEERING.—GRID ROTATION IN LÜDERS BANDS . . . . . *By T. Y. Thomas* 565

ENGINEERING.—A DISCUSSION OF THE LOAD DROP AND RELATED MATTERS ASSOCIATED WITH THE FORMATION OF A LÜDERS BAND . . . . . *By T. Y. Thomas* 572

GENETICS.—THE RELATION BETWEEN X-RAY DOSAGE AND THE FREQUENCY OF SIMULATED HEALING OF CHROMOSOME BREAKAGES IN *Drosophila Melanogaster* FEMALES . . . . . *By Irwin H. Herskowitz* 576

MATHEMATICS.—ON TORSION IN LIE GROUPS. . . . . *By Raoul Bott* 586

MATHEMATICS.—VARIATION EQUILIBRIUM AND PARETO OPTIMUM . . *By Gerard Debreu* 588

MATHEMATICS.—INTERDEPENDENCE OF THE YIELD CONDITION AND THE STRESS-STRAIN RELATIONS FOR PLASTIC FLOW. . . . . *By T. Y. Thomas* 593

MATHEMATICS.—A GENERALIZATION OF THE FRÉCHET DISTANCE OF TWO CURVES . . . . . *By A. J. Ward* 598

MATHEMATICS.—ESTIMATION OF THE COMPONENTS OF STOCHASTIC STRUCTURES . . . . . *By J. Wolfowitz* 602

PATHOLOGY & BACTERIOLOGY.—THE TEMPERATURE-PRESSURE-INHIBITOR RELATIONS OF BACTERIAL LUMINESCENCE IN VITRO. *By Bernard L. Strehler and Frank H. Johnson* 606

PHYSICS.—REPRESENTATIONS OF THE ANTICOMMUTATION RELATIONS . . . . . *By L. Gårding and A. Wightman* 617

PHYSICS.—REPRESENTATIONS OF THE COMMUTATION RELATIONS . . . . . *By L. Gårding and A. Wightman* 622

ZOÖLOGY.—NOTE ON A "PERMANENT" EXPERIMENTAL ALTERATION OF GENETIC CONSTITUTION IN A NATURAL POPULATION . . . *By Caryl P. Haskins and Edna F. Haskins* 627

N.A.S. ORGANIZATION. 637

Page
541
549
551
557
565
572
576
586
588
593
598
602
606
617
622
627
637